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FURTHER REMARKS ON THE TWO-DIMENSIONAL MOTION OF A COMPRESSIBLE FLUID

By S. D. DAYMOND (*Liverpool*)

[Received 5 June 1945]

IN connexion with the two-dimensional motion of a compressible fluid it has been shown by W. Tollmien, in a work referred to in a previous paper,* that there are in all just three cases in which orthogonal curvilinear coordinates ξ, η can be found which have the following property:

The components v_1, v_2 of the fluid velocity v along the tangents to the curves $\xi = \text{constant}$, $\eta = \text{constant}$ are functions of only one (say η) of these coordinates.

To each set of coordinates thus found there corresponds a particular integral of the hydrodynamical equations, and the object of this note is to give a direct derivation of these integrals, by application of the hodograph method, without the use of curvilinear coordinates. Just one of the integrals is non-trivial, and the main result obtained below is the appropriate relation expressing the correspondence between the plane of cartesian coordinates and that of the hodograph. The notation used here is that to be found in the previous article on the hodograph method, and numbers in parentheses are references to equations in that article.

The curves $\eta = \text{constant}$ must evidently be curves of constant speed (i.e. of constant λ), and there is the further restriction that along any one of them $\tan \beta = v_2/v_1$ is constant. Since $\tan(\theta - \beta) = y_\theta/x_\theta$ on any such curve, it follows that, throughout the region of flow,

$$\frac{x_\theta \sin \theta - y_\theta \cos \theta}{x_\theta \cos \theta + y_\theta \sin \theta}$$

is a function $T(\lambda)$ of λ only. Applications of (1.2), (1.3), and then (1.4), (1.5), lead respectively to

$$-\psi_\theta/\phi_\theta = \rho T \quad \text{and} \quad \phi_\lambda/\psi_\lambda = (\gamma+1)(1-\lambda)T/2\rho\gamma.$$

The only possibility therefore is that ψ and ϕ take the forms

$$\psi = \rho P(\lambda)Q(\theta), \quad \phi = R(\lambda)Q(\theta),$$

where, for convenience, the relative density ρ is included in the first

* S. D. Daymond, *Quart. J. of Math.* (Oxford), 16 (1945), 78-85.

form. It now follows from (1.4), (1.5), and the above ratios of partial derivatives that $Q \equiv e^{\alpha\theta}$, where α is a constant, and

$$\frac{d}{d\lambda}(\rho P) = \frac{1}{2}\alpha\rho R\lambda^{-1}, \quad \frac{dR}{d\lambda} = \alpha\rho P\nu'\lambda^{\frac{1}{2}},$$

so that ρP and R satisfy certain linear differential equations of the second order. Also $1/\rho T$ satisfies a first-order equation of the Riccati type.

With suitable choice of origin in the plane of x and y we have—see (1.2) and (1.3)—

$$(\alpha^2 + 1)x = \alpha\lambda^{-\frac{1}{2}}e^{\alpha\theta}\{(P + \alpha R)\cos\theta - (\alpha P - R)\sin\theta\},$$

$$(\alpha^2 + 1)y = \alpha\lambda^{-\frac{1}{2}}e^{\alpha\theta}\{(P + \alpha R)\sin\theta + (\alpha P - R)\cos\theta\}.$$

Thus the curves of constant speed are found to be equiangular spirals, the constant angle being $\cot^{-1}\alpha$.

The other two solutions correspond respectively to the values $\frac{1}{2}\pi$ and zero of β . In the former motion $x_\theta \cos\theta + y_\theta \sin\theta$ is zero, and consequently $\psi = A + B\theta$. The motion is that due to a 'source'—see (2.3). Similarly, the zero value of β corresponds to the motion arising from a 'vortex'. The streamlines and curves of constant speed are the same set of concentric circles, ψ being a function of λ only which is readily obtained from (1.6).

Author's note. Professor L. Rosenhead recently drew my attention to the existence of a work by F. Ringleb,* and also subsequently to the English translation† issued by the Ministry of Aircraft Production. The appropriate number of the above foreign publication seems to be at present not available, but I have now seen a copy of the translation. In this work will be found elaborate treatments of compressible-fluid motion arising from the source and vortex respectively; also the solution (3.4) of my previous paper receives special attention. It will be seen also that my condition (3.5) for singular points on the streamlines is identical with that for a 'flow-shock' (i.e. a point of infinite acceleration) as given by Ringleb, namely

$$v^2\psi_v^2 + \left(1 - \frac{v^2}{a^2}\right)\psi_\theta^2 = 0.$$

Thus 'envelope' in my paper and 'shock-line' in that of Ringleb are synonymous.

* *Zeits. f. angew. Math. u. Mech.* 20 (1940), 185–98.

† R. T. P. translation No. 1609.

THE DETERMINATION OF CAPACITY FOR TWO-DIMENSIONAL SYSTEMS OF CYLINDRICAL CONDUCTORS

By J. W. CRAGGS (*Military College of Science*)

[Received 5 June 1945]

THE usual method employed for two-dimensional problems in electrostatics is to seek a solution of Laplace's equation satisfying given boundary conditions. This procedure is satisfactory when the region of validity of Laplace's equation is the complete region between two closed curves, but, when three or more conductors are involved, the method breaks down, save in degenerate cases where one conductor completely separates the other two.

In the present paper I overcome this difficulty by using the distributions of charge on the n conductors as n functions to be determined from the constancy of potential on each conductor. The problem thus becomes one of determining a number of functions each of one independent variable instead of a single function (the potential) of two independent variables. For cylindrical conductors the potential at any point due to an arbitrary distribution of charge takes a simple form, and I therefore illustrate the method with reference to such conductors.

We shall constantly require the two following results.

1. Let the surface-density of charge $f(\phi)$ on a circular boundary of radius a be represented by a Fourier series

$$2af(\phi) = e \left[1 + \sum_{n=1}^{\infty} (a_n \cos n\phi + b_n \sin n\phi) \right], \quad (1)$$

so that the total charge

$$\int_0^{2\pi} af(\phi) d\phi = e; \quad (2)$$

then the potential V at the point (r, θ) is given, when $r \geq a$, by

$$\begin{aligned} V &= -a \int_{\theta}^{2\pi+\theta} f(\phi) \log[r^2 + a^2 - 2ra \cos(\phi - \theta)] d\phi \\ &= -2e \log r + e \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) (a/r)^n / n, \end{aligned} \quad (3)$$

by using the results*

$$\int_0^\pi \cos n\theta \log(1+A^2-2A \cos \theta) d\theta = \begin{cases} -\pi A^n/n & (A^2 \leq 1; n = 1, 2, \dots), \\ 0 & (A^2 \leq 1; n = 0). \end{cases}$$

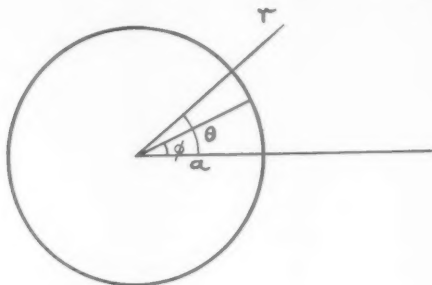


FIG. 1.

2. We now determine the value of this potential on a circular boundary with radius b and centre at the point $(c, 0)$. Then, at a point whose radius makes an angle ω with the line of centres,

$$V = -2e \log c + 2e \sum_{n=1}^{\infty} (b/c)^n \frac{\cos n\omega}{n} + e \sum_{p=1}^{\infty} \frac{(a/c)^p}{p} \sum_{n=0}^{\infty} {}^p B_n (b/c)^n (a_p \cos n\omega + b_p \sin n\omega), \quad (4)$$

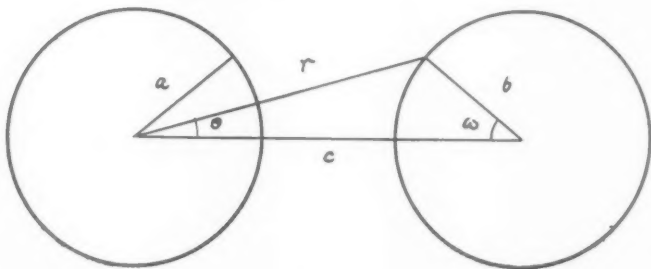


FIG. 2.

where ${}^p B_n \equiv p(p+1)\dots(p+n-1)/n!$ is the coefficient of x^n in $(1-x)^{-p}$. The result (4) follows from (3) by writing $r \sin \theta = b \sin \omega$, $r \cos \theta = c - b \cos \omega$ and expanding in terms of ω .

* J. Edwards, *The Integral Calculus* (2) (Macmillan, 1922), 306.

As an illustration and test of the method we consider first the well-known problem of the capacity of two parallel equal wires of radius a with centres a distance c apart.

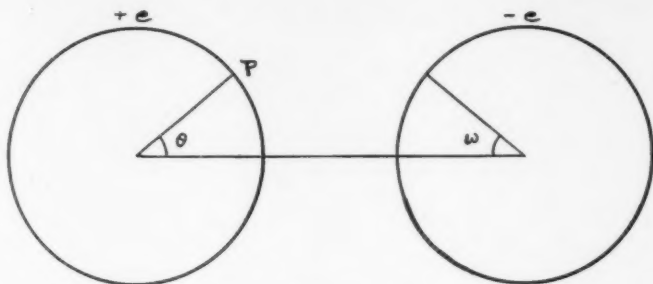


FIG. 3.

Assume charges $+e$ and $-e$ per unit length and potentials $\pm V$ on the wires, and measure the polar angles θ , ω as shown in Fig. 3. Then, from symmetry, the surface-densities of charge are $f(\theta)$ and $-f(\omega)$, where

$$2af(\theta) = e \left[1 + \sum_{n=1}^{\infty} a_n \cos n\theta \right].$$

The potential at P (Fig. 3) is given by

$$V = -2e \log a + \sum_{n=1}^{\infty} a_n \frac{\cos n\theta}{n} + 2e \log c - 2e \sum_{n=1}^{\infty} (a/c)^n \frac{\cos n\theta}{n} - e \sum_{p=1}^{\infty} \frac{(a/c)^p}{p} a_p \sum_{n=0}^{\infty} {}^p B_n (a/c)^n \cos n\theta, \quad (5)$$

by (3) and (4). Now the potential at P is independent of θ , and the capacity Q of the system is given by

$$\frac{1}{Q} = \frac{2V}{e};$$

$$\text{thus} \quad \frac{1}{2Q} + 2 \log \left(\frac{a}{c} \right) + \sum_{p=1}^{\infty} (a/c)^p \frac{a_p}{p} = 0 \quad (6)$$

$$\text{and} \quad \frac{a_n}{n} - \frac{2}{n} \left(\frac{a}{c} \right)^n - \sum_{p=1}^{\infty} {}^p B_n (a/c)^{n+p} \frac{a_p}{p} = 0 \quad (n = 1, 2, 3, \dots). \quad (7)$$

Eliminating the coefficients a_n/n and writing $a/c = d$ we have

$$\begin{vmatrix} 1/4Q + \log d & d & d^2 & d^3 & . & . \\ -d/1 & 1 - {}^1B_1 d^2 & -{}^2B_1 d^3 & -{}^3B_1 d^4 & . & . \\ -d^2/2 & -{}^1B_2 d^3 & 1 - {}^2B_2 d^4 & -{}^3B_2 d^5 & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{vmatrix} = 0. \quad (8)$$

It can easily be shown that the infinite determinant converges provided that $d < \frac{1}{2}$,* a condition which is clearly satisfied physically. Keeping one, two, three rows and columns we get

$$1/4Q = -\log d,$$

$$1/4Q = -\log d - \frac{d^2}{1-d^2},$$

$$1/4Q = -\log d - \frac{d^2}{2} \left(\frac{2+d^2-3d^4}{1-d^2-3d^4+d^6} \right)$$

respectively. Comparison with the usual form

$$\frac{1}{2Q} = \cosh^{-1} \frac{1}{2d}$$

shows that the convergence of the determinant is sufficiently rapid for computational purposes.

The application of the method to problems of three or more

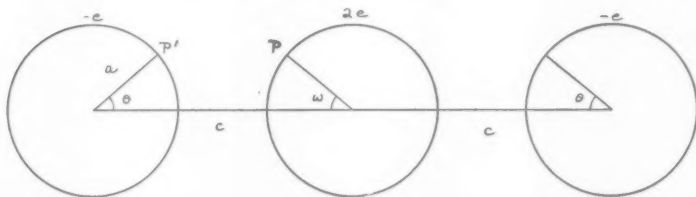


FIG. 4.

cylindrical conductors introduces no further difficulty. As a simple illustration I find the capacity of a system consisting of a charged wire of radius a placed symmetrically between two equal connected wires spaced $2c$ apart. Let the charge per unit length on the central wire be $2e$; then, from symmetry, that on each earthed wire is $-e$.

* Whittaker and Watson, *Modern Analysis* (Cambridge, 1920), 36.

Measure θ , ω as shown; then we may assume that the charge on the central conductor is $f(\omega)$, where

$$2\pi a f(\omega) = 2e \left[1 + \sum_{n=1}^{\infty} a_{2n} \cos 2n\omega \right],$$

and on an outer conductor $-g(\theta)$, where

$$2\pi a g(\theta) = e \left[1 + \sum_{n=1}^{\infty} b_n \cos n\theta \right].$$

The potential at P is now given by

$$\begin{aligned} V_1 &\equiv -4e \log a + 2e \sum_{n=1}^{\infty} a_{2n} \frac{\cos 2n\omega}{2n} + \\ &+ 2e \log c - 2e \sum_{n=1}^{\infty} (a/c)^n \frac{\cos n\omega}{n} - e \sum_{p=1}^{\infty} \frac{(a/c)^p}{p} b_p \sum_{n=0}^{\infty} {}^p B_n (a/c)^n \cos n\omega + \\ &+ 2e \log c - 2e \sum_{n=1}^{\infty} (-a/c)^n \frac{\cos n\omega}{n} - e \sum_{p=1}^{\infty} \frac{(a/c)^p}{p} b_p \sum_{n=0}^{\infty} {}^p B_n (-a/c)^n \cos n\omega \\ &\equiv -4e \log(a/c) - 2e \sum_{p=1}^{\infty} \frac{(a/c)^p}{p} b_p - \\ &- 2e \sum_{p=1}^{\infty} \left[(a/c)^{2n} \left(\frac{1}{n} + \sum_{p=1}^{\infty} {}^p B_{2n} (a/c)^p \frac{b_p}{p} \right) - \frac{a_{2n}}{2n} \right] \cos 2n\omega. \quad (9) \end{aligned}$$

From the identity in ω we obtain

$$V_1 = -4e \log(a/c) - 2e \sum_{p=1}^{\infty} (a/c)^p b_p / p \quad (10)$$

and

$$a_{2n}/2n = (a/c)^{2n} \left((1/n) + \sum_{p=1}^{\infty} {}^p B_n (a/c)^p b_p / p \right) \quad (n = 1, 2, 3, \dots). \quad (11)$$

Similarly, consideration of the potential at P' gives

$$\begin{aligned} V_2 &\equiv -4e \log c + 4e \sum_{n=1}^{\infty} (a/c)^n \frac{\cos n\theta}{n} + \\ &+ 2e \sum_{p=1}^{\infty} (a/c)^{2p} \frac{a_{2p}}{p} \sum_{n=0}^{\infty} {}^{2p} B_n (a/c)^n \cos n\theta + 2e \log a - e \sum_{n=1}^{\infty} b_n \frac{\cos n\theta}{n} + \\ &+ 2e \log 2c - 2e \sum_{n=1}^{\infty} (a/2c)^n \frac{\cos n\theta}{n} - e \sum_{p=1}^{\infty} (a/2c)^p \frac{b_p}{p} \sum_{n=0}^{\infty} {}^p B_n (a/2c)^n \cos n\theta \end{aligned}$$

$$\begin{aligned} &\equiv 2e \log(2a/c) + e \sum_{p=1}^{\infty} \left\{ \left(\frac{a}{c} \right)^{2p} \frac{a_{2p}}{p} - \left(\frac{a}{2c} \right)^p \frac{b_p}{p} \right\} - e \sum_{n=1}^{\infty} \left[\frac{2}{n} \left(\frac{a}{c} \right)^n \left(\frac{1}{2^n} - 2 \right) + \right. \\ &\quad \left. + \left(\frac{a}{c} \right)^n \sum_{p=1}^{\infty} \left\{ {}^p B_n \left(\frac{a}{2c} \right)^p \frac{b_p}{p} \frac{1}{2^n} - {}^{2p} B_n \left(\frac{a}{c} \right)^{2p} \frac{a_{2p}}{p} \right\} + \frac{b_n}{n} \right] \cos n\theta, \quad (12) \end{aligned}$$

leading to

$$\begin{aligned} V_2 &= 2e \log \left(\frac{2a}{c} \right) + e \sum_{p=1}^{\infty} \left\{ \left(\frac{a}{c} \right)^{2p} a_{2p} - \left(\frac{a}{2c} \right)^p b_p \right\} / p \\ &= 2e \log \left(\frac{2a}{c} \right) + e \sum_{p=1}^{\infty} \left[\frac{2}{p} \left(\frac{a}{c} \right)^{4p} \left(\frac{1}{p} + \sum_{q=1}^{\infty} {}^q B_{2p} \left(\frac{a}{c} \right)^q \frac{b_q}{q} \right) - \left(\frac{a}{2c} \right)^p \frac{b_p}{p} \right], \quad (13) \end{aligned}$$

and

$$\begin{aligned} &\left(\frac{a}{c} \right)^n \left[\frac{2}{n} \left(\frac{1}{2^n} - 2 \right) + \sum_{p=1}^{\infty} \left\{ {}^p B_n \left(\frac{a}{2c} \right)^p \frac{b_p}{p} \frac{1}{2^n} \right\} \right] + \frac{b_n}{n} = \sum_{p=1}^{\infty} {}^{2p} B_n \left(\frac{a}{c} \right)^{2p+n} a_{2p}/p \\ &= \sum_{p=1}^{\infty} {}^{2p} B_n 2 \left(\frac{a}{c} \right)^{4p+n} \left\{ \frac{1}{p} + \sum_{q=1}^{\infty} {}^q B_{2p} \left(\frac{a}{c} \right)^q \frac{b_q}{q} \right\} \quad (n = 1, 2, 3, \dots) \quad (14) \end{aligned}$$

by using (11) in each case.

Now, if we write $d = a/c$, the capacity Q is given by

$$\frac{1}{Q} = \frac{V_1 - V_2}{2e} = \log \left(\frac{1-d^4}{2d^3} \right) - \frac{1}{2} \sum_{p=1}^{\infty} \left[\frac{1}{(1+d^2)^p} + \frac{1}{(1-d^2)^p} - \frac{1}{2^p} \right] \frac{b_p}{p}, \quad (15)$$

where the coefficients b are given by

$$\frac{b_n}{n} - \frac{4\alpha_n}{n} d^n + \sum_{p=1}^{\infty} A_{pn} d^{p+n} \frac{b_p}{p} = 0 \quad (16)$$

$$\text{with} \quad A_{pn} = {}^p B_n \frac{1}{2^{n+p}} - 2 \sum_{q=1}^{\infty} {}^{2q} B_n {}^p B_{2q} d^{4q}, \quad (17)$$

so that $A_{pn} = \frac{p}{n} A_{np}$; and

$$\alpha_n = \frac{1}{2} \left[\frac{1}{(1+d^2)^n} + \frac{1}{(1-d^2)^n} - \frac{1}{2^n} \right]. \quad (18)$$

Then, eliminating b_p/p , we obtain

$$\begin{vmatrix} \frac{1}{4Q} - \frac{1}{4} \log \left(\frac{1-d^4}{2d^3} \right) & \alpha_1 d & \alpha_2 d^2 & . & . & . \\ -\frac{\alpha_1}{1} d & 1 + A_{11} d^2 & A_{21} d^3 & . & . & . \\ -\frac{\alpha_2}{2} d^2 & A_{12} d^3 & 1 + A_{22} d^4 & . & . & . \\ . & . & . & . & . & . \end{vmatrix} = 0. \quad (19)$$

From this determinant the capacity can be calculated without excessive labour provided that the conductors are not too close together.

In conclusion I wish to thank Mr. C. J. Tranter for much helpful criticism and assistance in preparing the manuscript.

THE CAPACITY OF TWO-DIMENSIONAL SYSTEMS OF CONDUCTORS AND DIELECTRICS WITH CIRCULAR BOUNDARIES

By J. W. CRAGGS and C. J. TRANTER

(*Military College of Science*)

[Received 18 July 1945]

In a recent paper by one of us* a method was given for the solution of two-dimensional electrostatic problems involving cylindrical conductors. The method is here extended to deal with problems in which different dielectrics are present, the boundaries again being circular cylinders. Two important practical cases of transmission lines are discussed by way of example.

In (A) the field due to a system of conductors was expressed in terms of the assumed charge-distributions on them, these distributions then being determined from the constancy of the potential over the surfaces. To allow for the effect of different homogeneous dielectrics we use the well-known principle of placing a fictitious surface-charge on each dielectric interface. The distribution of this charge is then chosen to preserve the continuity of the normal component of polarization over the common boundary of two dielectrics.

Before illustrating the method by reference to particular examples, we state the following results.

With polar coordinates, let the surface-density of charge on $r = a$ be

$$f(\theta) = \frac{e}{2\pi a} \left[a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \right]; \quad (1)$$

then the potential at (r, ϕ) is

$$V = \begin{cases} -2ea_0 \log a + e \sum_{n=1}^{\infty} \frac{(a_n \cos n\phi + b_n \sin n\phi) \left(\frac{r}{a}\right)^n}{n} & (r \leq a) \\ -2ea_0 \log r + e \sum_{n=1}^{\infty} \frac{(a_n \cos n\phi + b_n \sin n\phi) \left(\frac{a}{r}\right)^n}{n} & (r \geq a) \end{cases}. \quad (2)$$

* J. W. Craggs, 'The determination of capacity for two-dimensional systems of cylindrical conductors': see above pp. 131-7. Subsequently referred to as (A).

Let ρ, ω be polar coordinates with pole at $r = c, \theta = 0$ and initial line $\theta = 0$. Then the potential at (ρ, ω) is*

$$V = -2ea_0 \log c + 2ea_0 \sum_{n=1}^{\infty} \left(-\frac{\rho}{c} \right)^n \frac{\cos n\omega}{n} + \\ + e \sum_{p=1}^{\infty} \left(\frac{a}{c} \right)^p \frac{1}{p} \sum_{n=0}^{\infty} {}^p B_n \left(-\frac{\rho}{c} \right)^n (a_p \cos n\omega - b_p \sin n\omega) \\ (c > a; \rho < c-a), \quad (3)$$

$$V = -2ea_0 \log a + e \sum_{p=1}^{\infty} \left(\frac{c}{a} \right)^p \frac{1}{p} \sum_{n=0}^{\infty} {}^p C_n \left(\frac{\rho}{c} \right)^n (a_p \cos n\omega + b_p \sin n\omega) \\ (c < a; \rho < a-c \text{ and } \rho < c), \quad (4)$$

$$V = -2ea_0 \log \rho + 2ea_0 \sum_{n=1}^{\infty} \left(-\frac{c}{\rho} \right)^n \frac{\cos n\omega}{n} + \\ + e \sum_{p=1}^{\infty} \left(\frac{a}{\rho} \right)^p \frac{1}{p} \sum_{n=0}^{\infty} {}^p B_n \left(-\frac{c}{\rho} \right)^n \{a_p \cos(p+n)\omega + b_p \sin(p+n)\omega\} \\ (\rho > c+a). \quad (5)$$

Of these results, (2) and (3) have been obtained in (A), and (4) and (5) can be obtained similarly.

Our first illustration is the problem of the capacity of twin cable, each conductor being surrounded by a concentric dielectric sheath. We have previously discussed this problem elsewhere,† using the method of normal functions, but the present method gives the solution in a form more satisfactory for computation. The problem may be stated, in terms of the coordinates defined above, as follows: two conductors $r = a, \rho = a$ at potentials $+V_0, -V_0$ are surrounded by sheaths $a < r < b, a < \rho < b$, of dielectric constant K_1 , the region $r > b, \rho > b$ being of dielectric constant K_2 .‡

Let the charge-densities on the conductors be $K_1 f(\theta), -K_1 f(\pi - \omega)$ respectively, where

$$2\pi a f(\theta) = e \left(1 + \sum_{n=1}^{\infty} a_n \cos n\theta \right), \quad (6)$$

so that the total charges are $\pm K_1 e$. Then the potential V can be

* ${}^p B_n$ denotes the coefficient of x^n in $(1-x)^{-p}$.

† J. W. Craggs and C. J. Tranter, 'The capacity of twin cable—II': *Quart. J. of Applied Math.* (3) 380.

‡ We assume here that $b < \frac{1}{2}c$, so that the sheaths are separate.

obtained by using charge-densities $f(\theta)$, $-f(\pi-\omega)$, with dielectric constants unity, where $a < r < b$, $a < \rho < b$, and $1/K$ elsewhere, where $K = K_1/K_2$.

We now replace the dielectric $1/K$ by surface-charges $g(\theta)$, $-g(\pi-\omega)$ on $r = b$, $\rho = b$, where

$$2\pi b g(\theta) = e \left(b_0 + \sum_{n=1}^{\infty} b_n \cos n\theta \right). \quad (7)$$

The dielectric boundary conditions then become

$$\left. \begin{aligned} \left(\frac{\partial V}{\partial r} \right)_{r=b-0} &= \frac{4\pi}{1-K} g(\theta) \\ \left(\frac{\partial V}{\partial \rho} \right)_{\rho=b-0} &= -\frac{4\pi}{1-K} g(\pi-\omega) \end{aligned} \right\}. \quad (8)$$

We now determine the coefficients a_n , b_n in (6) and (7) from the boundary condition (8) and the condition $V \equiv -V_0$ at $\rho = a$.

For $a < \rho < b$, the results (2) and (3) give

$$\begin{aligned} V = 2e \log \rho - e \sum_{n=1}^{\infty} \left(-\frac{a}{\rho} \right)^n \frac{a_n \cos n\omega}{n} + 2eb_0 \log b - \\ - e \sum_{n=1}^{\infty} \left(-\frac{\rho}{b} \right)^n \frac{b_n \cos n\omega}{n} - 2e \log c + 2e \sum_{n=1}^{\infty} \left(-\frac{\rho}{c} \right)^n \frac{\cos n\omega}{n} + \\ + e \sum_{p=1}^{\infty} \left(\frac{a}{c} \right)^p \frac{a_p}{p} \sum_{n=0}^{\infty} {}^p B_n \left(-\frac{\rho}{c} \right)^n \cos n\omega - \\ - 2eb_0 \log c + 2eb_0 \sum_{n=1}^{\infty} \left(-\frac{\rho}{c} \right)^n \frac{\cos n\omega}{n} + \\ + e \sum_{p=1}^{\infty} \left(\frac{b}{c} \right)^p \frac{b_p}{p} \sum_{n=0}^{\infty} {}^p B_n \left(-\frac{\rho}{c} \right)^n \cos n\omega. \end{aligned} \quad (9)$$

Differentiating with respect to ρ , putting $\rho = b$, using (8), and equating coefficients, we have

$$b_0 = K-1, \quad (10)$$

$$\begin{aligned} \frac{K+1}{K-1} b_n = 2K \left(\frac{b}{c} \right)^n + \left(\frac{a}{b} \right)^n a_n + n \left(\frac{b}{c} \right)^n \sum_{p=1}^{\infty} \frac{{}^p B_n}{p} \left\{ \left(\frac{a}{c} \right)^p a_p + \left(\frac{b}{c} \right)^p b_p \right\} \\ (n = 1, 2, \dots). \end{aligned} \quad (11)$$

Putting $\rho = a$, $V = -V_0$ in (9), we obtain similarly

$$-\frac{V_0}{e} = 2 \log \left(\frac{a}{c} \right) + 2(K-1) \log \left(\frac{b}{c} \right) + \sum_{p=1}^{\infty} \frac{1}{p} \left\{ \left(\frac{a}{c} \right)^p a_p + \left(\frac{b}{c} \right)^p b_p \right\}, \quad (12)$$

$$0 = \frac{a_n}{n} + \left(\frac{a}{b} \right)^n \frac{b_n}{n} - \frac{2K}{n} \left(\frac{a}{c} \right)^n - \left(\frac{a}{c} \right)^n \sum_{p=1}^{\infty} \frac{{}^p B_n}{p} \left\{ \left(\frac{a}{c} \right)^p a_p + \left(\frac{b}{c} \right)^p b_p \right\} \\ (n = 1, 2, \dots). \quad (13)$$

Eliminating the series-term from (11) and (13) gives

$$\frac{2}{K-1} b_n = \left\{ \left(\frac{a}{b} \right)^n + \left(\frac{b}{a} \right)^n \right\} a_n. \quad (14)$$

We now substitute for a_n in (12) and (13) and write Q for the capacity, i.e.

$$Q = \frac{K_1 e}{2V_0}.$$

Then

$$\frac{K_1}{2Q} + 2 \log \left(\frac{a}{c} \right) + 2(K-1) \log \left(\frac{b}{c} \right) + \\ + \frac{1}{K-1} \sum_{p=1}^{\infty} \frac{(K+1)a^{2p} + (K-1)b^{2p}}{a^{2p} + b^{2p}} \left(\frac{b}{c} \right)^p \frac{b_p}{p} = 0, \quad (15) \\ - \frac{2K}{n} \left(\frac{a}{c} \right)^n + \frac{1}{K-1} \frac{(K-1)a^{2n} + (K+1)b^{2n}}{a^{2n} + b^{2n}} \left(\frac{a}{b} \right)^n \frac{b_n}{n} - \\ - \frac{1}{K-1} \left(\frac{a}{c} \right)^n \sum_{p=1}^{\infty} {}^p B_n \frac{(K+1)a^{2p} + (K-1)b^{2p}}{a^{2p} + b^{2p}} \left(\frac{b}{c} \right)^p \frac{b_p}{p} = 0 \\ (n = 1, 2, \dots). \quad (16)$$

Eliminating $\frac{b_p}{p(K-1)(a^{2p} + b^{2p})}$ and writing

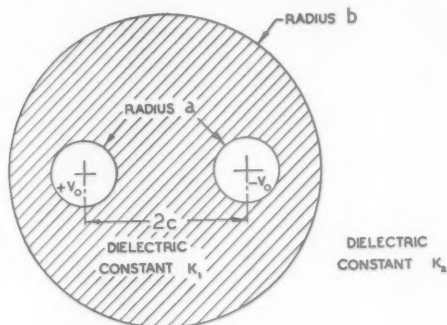
$$\left. \begin{aligned} \alpha_p &= (K+1)a^{2p} + (K-1)b^{2p} \\ \beta_p &= (K-1)a^{2p} + (K+1)b^{2p} \end{aligned} \right\}, \quad (17)$$

we have

$$\left| \begin{array}{cccc} \frac{K_1}{2Q} + 2 \log \left(\frac{a}{b} \frac{b}{c} \right)^K & \alpha_1 \left(\frac{b}{c} \right) & \alpha_2 \left(\frac{b}{c} \right)^2 & \dots \\ -2K \left(\frac{b}{c} \right) & \beta_1 - {}^1 B_1 \alpha_1 \left(\frac{b}{c} \right)^2 & -{}^2 B_1 \alpha_2 \left(\frac{b}{c} \right)^3 & \dots \\ -\frac{2K}{2} \left(\frac{b}{c} \right)^2 & -{}^1 B_2 \alpha_1 \left(\frac{b}{c} \right)^3 & \beta_2 - {}^2 B_2 \alpha_2 \left(\frac{b}{c} \right)^4 & \dots \\ \dots & \dots & \dots & \dots \end{array} \right| = 0. \quad (18)$$

The convergence of this determinant can easily be established, and satisfactory numerical approximations can be obtained by keeping only the first few rows and columns.

A second common type of cable is that shown in the diagram, where two equal wires are embedded in a homogeneous cylindrical dielectric which may or may not be contained in an earthed conducting sheath. We consider here only the case of balanced wires (potentials $\pm V_0$), and this solution we believe to be new.



We assume charge-distributions $K_1 f(\theta)$, $-K_1 f(\pi - \omega)$ over the conductors, $f(\theta)$ being given by (6). In view of the symmetry of the problem, the fictitious charge on the dielectric surface can be taken as $g(\phi)$, where

$$2\pi b g(\phi) = e \sum_{n=0}^{\infty} b_{2n+1} \cos(2n+1)\phi. \quad (19)$$

The angles θ , ω , ϕ are measured anti-clockwise at the centres of the respective circles.

Using (2)–(5) and proceeding as before, we obtain the condition on the dielectric boundary as

$$\begin{aligned} \frac{2}{1-K} \sum_{n=0}^{\infty} b_{2n+1} \cos(2n+1)\phi &= \frac{b}{e} \left(\frac{\partial V}{\partial r} \right)_{r=b} = 4 \sum_{n=0}^{\infty} \left(\frac{c}{b} \right)^{2n+1} \cos(2n+1)\phi - \\ &- \sum_{p=1}^{\infty} \left(\frac{a}{b} \right)^p \frac{a_p}{p} \sum_{n=0}^{\infty} {}^p B_n \left(\frac{c}{b} \right)^n \{ (-1)^n - (-1)^p \} (p+n) \cos(p+n)\phi + \\ &+ \sum_{n=0}^{\infty} b_{2n+1} \cos(2n+1)\phi \quad (0 \leq \phi \leq 2\pi), \end{aligned} \quad (20)$$

leading to

$$\left(\frac{1+K}{1-K}\right)b_{2n+1} = 2\left(\frac{c}{b}\right)^{2n+1} \left\{ 2 + (2n+1) \sum_{p=1}^{2n+1} {}^pB_{2n+1-p} \left(-\frac{a}{c}\right)^p \frac{a_p}{p} \right\} \quad (n = 0, 1, 2, \dots), \quad (21)$$

where $K = K_1/K_2$.

The condition on the right-hand conductor is

$$\begin{aligned} -\frac{V_0}{e} = & -2 \log 2c + 2 \sum_{n=1}^{\infty} \left(-\frac{a}{2c}\right)^n \frac{\cos n\omega}{n} + \\ & + \sum_{p=1}^{\infty} \left(\frac{a}{2c}\right)^p \frac{a_p}{p} \sum_{n=0}^{\infty} {}^pB_n \left(-\frac{a}{2c}\right)^n \cos n\omega + 2 \log a - \sum_{n=1}^{\infty} (-1)^n \frac{a_n \cos n\omega}{n} + \\ & + \sum_{p=0}^{\infty} \left(\frac{c}{b}\right)^{2p+1} \frac{b_{2p+1}}{2p+1} \sum_{n=0}^{2p+1} {}^{2p+1}C_n \left(\frac{a}{c}\right)^n \cos n\omega \quad (0 \leq \omega \leq 2\pi), \end{aligned} \quad (22)$$

whence

$$-\frac{V_0}{e} = 2 \log \left(\frac{a}{2c}\right) + \sum_{p=1}^{\infty} \left(\frac{a}{2c}\right)^p \frac{a_p}{p} + \sum_{p=0}^{\infty} \left(\frac{c}{b}\right)^{2p+1} \frac{b_{2p+1}}{2p+1}, \quad (23)$$

and

$$\begin{aligned} 0 = & (-1)^{n+1} \frac{a_n}{n} + \frac{2}{n} \left(-\frac{a}{2c}\right)^n + \left(-\frac{a}{2c}\right)^n \sum_{p=1}^{\infty} {}^pB_n \left(\frac{a}{2c}\right)^p \frac{a_p}{p} + \\ & + \left(\frac{a}{c}\right)^n \sum_{2p+1=n}^{\infty} {}^{2p+1}C_n \left(\frac{c}{b}\right)^{2p+1} \frac{b_{2p+1}}{2p+1} \quad (n = 1, 2, \dots), \end{aligned} \quad (24)$$

the first term of the last summation being given by $p = \frac{1}{2}n$ when n is even.

Substitution for b_{2p+1} from (21) yields

$$\frac{K_1}{2Q} + 2 \log \left(\frac{a}{2c}\right) - 2 \left(\frac{K-1}{K+1}\right) \log \left(\frac{b^2+c^2}{b^2-c^2}\right) + \sum_{p=1}^{\infty} \alpha_p \left(-\frac{a}{c}\right)^p \frac{a_p}{p} = 0, \quad (25)$$

$$(-1)^n \frac{a_n}{n} - \frac{2}{n} \left(\frac{a}{c}\right)^n \alpha_n + \left(\frac{a}{c}\right)^n \sum_{p=1}^{\infty} A_{pn} \left(-\frac{a}{c}\right)^p \frac{a_p}{p} = 0, \quad (26)$$

where Q is the capacity and

$$\left. \begin{aligned} \alpha_p = & \left(-\frac{1}{2}\right)^p - \left(\frac{K-1}{K+1}\right) \left\{ \left(\frac{c^2}{b^2-c^2}\right)^p - \left(\frac{-c^2}{b^2+c^2}\right)^p \right\} \\ A_{pn} = & -{}^pB_n \left(-\frac{1}{2}\right)^{p+n} + 2 \left(\frac{K-1}{K+1}\right) \sum_q^{\infty} {}^{2q+1}C_n {}^pB_{2q+1-p} \left(\frac{c}{b}\right)^{4q+2} \end{aligned} \right\}, \quad (27)$$

the \sum_q^∞ signifying that q runs through integer values between $\frac{1}{2}(n-1)$ or $\frac{1}{2}(p-1)$, whichever is the greater, and infinity.

Finally, eliminating $(-1)^p a_p/p$ from (25) and (26), we obtain

$$\left| \begin{array}{ccccccc} \frac{K_1}{4Q} + \log\left(\frac{a}{2c}\right) - \left(\frac{K-1}{K+1}\right) \log\left(\frac{b^2+c^2}{b^2-c^2}\right) & \alpha_1\left(\frac{a}{c}\right) & \alpha_2\left(\frac{a}{c}\right)^2 & . & . & . & . \\ & -\frac{\alpha_1}{1}\left(\frac{a}{c}\right) & 1+A_{11}\left(\frac{a}{c}\right)^2 & A_{21}\left(\frac{a}{c}\right)^3 & . & . & . \\ & -\frac{\alpha_2}{2}\left(\frac{a}{c}\right)^2 & A_{12}\left(\frac{a}{c}\right)^3 & 1+A_{22}\left(\frac{a}{c}\right)^4 & . & . & . \\ . & . & . & . & . & . & . \end{array} \right| = 0. \quad (28)$$

As before, satisfactory numerical approximations can be obtained by retaining only a few terms of this determinant. It is worth noticing that $nA_{pn} = pA_{np}$ and can easily be expressed in concise form for small p, n .

The result for screened cable can be obtained as the limiting case, $K_2 \rightarrow \infty$, i.e. $K = 0$.

NON-INTEGRAL FUNCTIONAL POWERS

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1. Introduction

IN a recent study* of functional products and powers, it was found that under certain conditions one function played the part of a non-integral power of another function. In the present paper we give formal definitions of such powers and their elementary properties, and we define a *power set* of functions of a single real variable, every member of which is a power (integral or non-integral) of every other member. For existence theorems it is sufficient to quote certain theorems given in the papers mentioned above.

2. Definitions

We consider only functions which are continuous and strictly increasing in some interval† $\langle a, b \rangle$. The identity function is denoted by I , so that $I(x) \equiv x$.

Functions f, g intersect at x_0 if $f(x_0) = g(x_0)$.

A *node* of f is a number satisfying $f(x) = x$, i.e. an intersection of f and I .

A function f is *complete* in $\langle a, b \rangle$ if it is continuous and strictly increasing and if a and b are nodes of f . If f is complete in $\langle a, b \rangle$ but has no node in the open interval (a, b) , then we say that f is c.n.f. in $\langle a, b \rangle$, this being written for 'complete and node-free'.

The *functional product* of f and g is $f\{g(x)\}$, which will be written $fg(x)$, or simply fg . If f and g are continuous and strictly increasing, then so also are their inverses f^{-1} , g^{-1} , and their product fg , and the inverse of this product is $g^{-1}f^{-1}$.

Integral *functional powers* of f are defined by the relations

$$f^0 = I, \quad f^n = ff^{n-1}, \quad f^{-n} = (f^{-1})^n, \quad (n = 2, 3, \dots). \quad (1)$$

Clearly, for all integers p, q ,

$$f^p f^q = f^{p+q}. \quad (2)$$

Functions f, g *commute* if $fg = gf$ in some interval. For example,

* A. G. Walker, 'Commutative Functions (I) and (II)': see above, pp. 65-92. These papers will be referred to as C.F. I and C.F. II.

† An angular bracket denotes closure.

f^p commutes with f^q for any integers p, q . Elementary properties of commutative functions were given in C.F. I, § 4.

The following properties are easily verified, it being understood here and later that p, q take only integral values, positive, zero, and negative.

If a number of functions are each complete in an interval $\langle a, b \rangle$, then their powers and products also have this property.

If f is c.n.f. in $\langle a, b \rangle$, then f^p , for every non-zero p , also has this property.

If f is c.n.f. in $\langle a, b \rangle$ and $f > I$, this being understood to hold in the open interval (a, b) , then $f^p \geq f^q$ according as $p \geq q$. In particular, if $f > I$, then $f^{-1} < I$, and vice versa.

3. Non-integral functional powers

From our definition (1) we see that each integral power f^p of a given function f is unique. We shall find, however, that this property does not extend to non-integral powers, and for this reason we introduce a symbol \doteq which will be properly defined later; thus $g \doteq f^\lambda$ can be read ' g behaves like f^λ '. When f and a non-integral λ are given, g is not unique. Also the relationship denoted by \doteq is not transitive, i.e. if $g \doteq f^\lambda$ and $h \doteq g^\mu$, it does not necessarily follow, in general, that $h \doteq f^\nu$ for some ν . Later, however, we shall restrict ourselves to certain sets of functions (power sets) in which the relationship is transitive.

In particular cases, the form of f^p as a function of p indicates a reasonable definition of f^λ for non-integral λ . Consider, for example, $f(x) \equiv x^k$. By direct calculation we find $f^p(x) \equiv x^{kp}$, which leads to the definition $f^\lambda(x) \equiv x^{k\lambda}$ for all λ . Another example is $f(x) \equiv mx + k$, in which case we find

$$f^\lambda(x) \equiv m^\lambda x + \frac{1-m^\lambda}{1-m} k.$$

It can be verified that in both these examples the index law (2) is satisfied by all powers of f .

More generally, it has been shown* that any function f which is c.n.f. in some interval $\langle a, b \rangle$ can be expressed in the *canonical form* $\psi^{-1}\alpha\psi$, where ψ is a continuous strictly increasing function such that $\psi(a) = 0$, $\psi(b) = \infty$, and α is a positive constant. It follows at once that $f^p = \psi^{-1}\alpha^p\psi$ for all p , and a reasonable definition of f^λ is

* This theorem, essentially due to G. J. Whitrow, was proved in C.F. I, § 3.

therefore $\psi^{-1}\alpha^\lambda\psi$ for all non-integral λ . With this definition we find

$$f^\lambda f^\mu = \psi^{-1}\alpha^\lambda\psi\psi^{-1}\alpha^\mu\psi = \psi^{-1}\alpha^{\lambda+\mu}\psi = f^{\lambda+\mu},$$

so that (2) is satisfied for all indices.

This canonical form illustrates the fact, mentioned above, that f^λ is not unique when f and λ are given, λ being non-integral. For it has been shown (C.F. I, § 3) that ψ is not uniquely determined by f and α , and we therefore have a set of functional powers corresponding to each suitable ψ . If f^λ is given as above by one ψ and f^μ by another, then, in general, $f^\lambda f^\mu$ does not behave like $f^{\lambda+\mu}$.

4. Essential properties of functional powers

Before giving a formal definition, we shall first state those properties which we think functional powers should reasonably possess. As already mentioned, we confine ourselves to continuous strictly increasing functions. We can also suppose that the *base function*, say f , whose powers we are considering, is c.n.f. in an interval $\langle a, b \rangle$, and it is sufficient for us to consider this interval alone, since functional properties in two such intervals are independent.* We further suppose that $f > I$ in (a, b) , corresponding results for $f < I$ being found by taking inverses.

It is clear from the index law (2) and the study of canonical forms that our main problem concerns not just one function (apart from f) but a whole set, including, for consistency, f^p for all integers p . The elementary properties of such a set which we regard as desirable are analogous to properties of f^p and are as follows:

- (i) every function of the set is c.n.f. in $\langle a, b \rangle$;
- (ii) if $g \doteq f^\lambda$ and $h \doteq f^\mu$, then $g^p h^q \doteq f^{p\lambda+q\mu}$ for all p, q ;
- (iii) if g, h are as in (ii), then $g \gtrless h$ according as $\lambda \gtrless \mu$.

These properties are certainly possessed by all integral powers of f , and by those special sets of functions considered in § 3. From (ii) with $p = q = 1$, we deduce that $gh = hg$; our set of functions is thus commutative. From results given in C.F. I and II it follows that the above conditions are satisfied by *related* and *semi-related* sets of functions. From a property of semi-related sets we at once deduce that *not every set of functional powers is such that its members can be expressed in canonical form simultaneously*.

* The justification for these assumptions is to be found in C.F. II, Appendix, and C.F. I, § 5.

In the next section we give a sufficient set of conditions which, when satisfied by a number of functions, implies the existence of relationships having the above properties.

5. Power sets

DEFINITION. *A set P of functions is a 'power set' in an interval $\langle a, b \rangle$ if*

- (i) *every member of P is complete in $\langle a, b \rangle$;*
- (ii) *every member of P commutes with every other member;*
- (iii) *every integral power and every product of integral powers of members of P is either node-free in $\langle a, b \rangle$ or is the identity function.*

From elementary properties of commutative functions it follows that, if g, h are any two members of P , then g^p (for all integral p) and gh also satisfy the above conditions in relation to P . The set P can therefore be augmented to include all integral powers of every member and the functional product of every pair. Assuming that this has been done, then, if g, h are any two members of P , so also are g^p and gh .

We now choose for base function any member of P , say f , other than the identity. This function is node-free in (a, b) by (iii), and we suppose for convenience that $f > I$. Our argument can be repeated, with certain inequalities reversed, when $f < I$, and it is easily seen how results are affected by this change.

We proceed to show that with each member g of P is associated a real number λ , which will be called the *index of g with respect to f* . It occurs as a section (L, R) of the rational numbers p/n ($n > 0$), the numbers in L being those for which $g^n > f^p$, and the numbers in R being those for which $g^n \leq f^p$. That every rational belongs either to L or to R follows from (iii), which implies that f^p and g^n , for any p, n , do not intersect without coinciding. We also see, from the order properties of integral functional powers described in § 2, that every rational in L is less than every rational in R . It only remains, therefore, to show that the third Dedekind condition is satisfied, i.e. that both L and R exist.

Suppose that $g > f$ in (a, b) ; then L contains 1. If $a < x_0 < b$, then, since f is c.n.f. in $\langle a, b \rangle$ and $f > I$, we have $f^p(x_0) \rightarrow b$ as $p \rightarrow \infty$. From (i), $a < g(x_0) < b$, and there is therefore a positive p such that $f^p(x_0) > g(x_0)$ and hence $f^p > g$. This p therefore belongs to R , so that both L and R exist in this case. A similar argument proves

their existence in the case $g < f$, including the case $g < I$. All the Dedekind conditions are thus satisfied, and a number λ is defined by the above section.

If, in a given power set, λ is the index of g with respect to f , we shall write $g \doteq f^\lambda$.

From the definition of index and the order properties of integral functional powers we have at once:

THEOREM 5.1. *If $g \doteq f^\lambda$, then $g^{-1} \doteq f^{-\lambda}$.*

The following consequences are immediate.

If λ is integral, say p , then g is f^p as previously defined.

If λ is fractional, say p/n , then $f^p = g^n$ throughout (a, b) . Since also f, g commute by (iii) above, they are *rationally related* (C.F. I, § 7), and there is a function ϕ such that $f = \phi^n, g = \phi^p$.

If λ is irrational, then for all p, q not both zero, f^p, g^q do not intersect in (a, b) . Since f, g commute, it follows that they are either *irrationally related* or *semi-related* (C.F. I, § 8), and in the former case they can be expressed in canonical form simultaneously, i.e. as in § 3 with the same ψ .

6. Index properties of a power set

We shall now prove that a power set has the properties described in § 4. The main theorem is the following:

THEOREM 6.1. *If $g \doteq f^\lambda$ and $h \doteq f^\mu$, then $gh \doteq f^{\lambda+\mu}$.*

We use the lemma:

LEMMA. *If continuous, strictly increasing functions θ, ϕ commute and satisfy $\theta^m > \phi^m$ in (a, b) , m being a positive integer, then $\theta > \phi$ in this interval.*

The proof of the lemma is clear. To prove the theorem, consider any rational p/n ($n > 0$). Suppose first that $p/n < \lambda + \mu$; then we can find $m > 0$ such that $mp = p' + p''$, where $p' < mn\lambda$ and $p'' < mn\mu$. From the definitions of λ and μ we therefore have $g^{mn} > f^{p'}$ and $h^{mn} > f^{p''}$, whence

$$g^{mn}h^{mn} > g^{mn}f^{p''} > f^{p'}f^{p''} = f^{mp},$$

since g^{mn} is strictly increasing. Since g, h commute,

$$g^{mn}h^{mn} = \{(gh)^n\}^m,$$

and we have

$$\{(gh)^n\}^m > (f^p)^m.$$

Therefore, by the lemma $(gh)^n > f^p$.

(3)

The function gh belongs to P and therefore has an index, say ν , with respect to f . From (3) it follows that ν satisfies $n\nu > p$ for all p, n such that $p/n < \lambda + \mu$ ($n > 0$), and it can similarly be proved that $n\nu < p$ when $p/n > \lambda + \mu$. Hence, $\nu = \lambda + \mu$, and the theorem is proved.

From this theorem and the definition of index it is clear that, if $\lambda + \mu$ is an integer, say p , then gh is f^p as previously defined. In particular, if $g \doteq f^\lambda$ and $h \doteq f^{-\lambda}$, then $gh = I$, and h is therefore g^{-1} . This is the converse of Theorem 5.1, and is contained in the following:

THEOREM 6.2. *If $g \doteq f^\lambda$ and $h \doteq f^{-\lambda}$, then $g = h$.*

For we have $h^{-1} \doteq f^{-\lambda}$ whence $gh^{-1} = I$.

By repeated application of Theorems 5.1 and 6.1 we have:

THEOREM 6.3. *If members f, g, h , of a power set are such that $g \doteq f^\lambda$ and $h \doteq f^\mu$, then $g^p h^q \doteq f^{p\lambda + q\mu}$ for all integers p, q .*

Turning to (iii) of § 4 we see that the case of equality is covered by Theorem 6.2. Suppose now that $g \doteq f^\lambda$, $h \doteq f^\mu$, and that $\lambda > \mu$. Then there are integers p, n such that $n\lambda > p$ and $n\mu < p$, and from the definition of index we have

$$g^n > f^p, \quad h^n \leq f^p.$$

Hence

$$g^n > h^n,$$

and from the lemma at the beginning of this section, $g > h$. Similarly, $\lambda < \mu$ implies $g < h$, and we have:

THEOREM 6.4. *If, in a power set, $g \doteq f^\lambda$ and $h \doteq f^\mu$, then $g \geq h$ according as $\lambda \geq \mu$.*

It has been remarked that the base function can be any member of P other than the identity. To observe the effect of a change of base, say from f to g where $f > I$ and $g > I$, suppose $g \doteq f^\lambda$, $\lambda > 0$, and consider $h = f^\mu$. For a rational p/n less than μ/λ ($n > 0$) we have $n\mu > p\lambda$ and hence, by the above theorems, $h^n > g^{p\lambda}$. Similarly, $h^n \leq g^{p\lambda}$ when $p/n \geq \mu/\lambda$. Hence:

THEOREM 6.5. *If, in a power set, $g \doteq f^\lambda$ and $h \doteq f^\mu$, then $h \doteq g^{\mu/\lambda}$.*

Although this theorem has been proved only for bases greater than the identity, it can be shown to be true generally. For a base function less than the identity, some of the inequalities in the definition and theorems on indices are reversed, but all theorems of equality remain unaltered.

COROLLARY 1. *If $g \doteq f^\lambda$, then $f \doteq g^{1/\lambda}$.*

COROLLARY 2. *If $g \doteq f^\lambda$ and $h \doteq g^\nu$, then $h \doteq f^{\lambda\nu}$.*

7. Classification of power sets

In a power set P , let Λ be the set of indices of members of P with respect to a base f . Remembering that Λ certainly contains all the integers, and that, if λ, μ are members of Λ , then $\lambda \pm \mu$ are also members, we see that there are various possibilities.

I. Λ may consist only of the integers. In this case, the members of P are all integral functional powers of one member; such a P will be called an *integral power set*.

II. Λ may consist only of discrete rationals. From the fact concerning $\lambda \pm \mu$ mentioned above it is easily seen that in this case all members of Λ are integral multiples of one member, say λ . If therefore $g \doteq f^\lambda$, then all members of P are integral powers of g , and P is an *integral power set*.

III. If all members of Λ are rationals but Λ is not discrete, then clearly Λ is everywhere-dense.* In this case we shall call P a *rational power set*. (Strictly, it should be 'rational non-integral'.)

In a rational power set, every member is rationally related to every other member (C.F. I, § 7), but P may or may not be a *related set*, i.e. such that all members are expressible simultaneously in canonical form $\psi^{-1}\alpha^\lambda\psi$. In the alternative case, the set will be described as *semi-related*, because it has many of the properties of semi-related functions (C.F. II). The existence of related rational power sets is clear; the existence of semi-related rational power sets was indicated in C.F. I at the end of § 12, and it is hoped that such sets will be described in a later paper.

The definition of a rational power set is clearly independent of the choice of base function.

IV. If Λ contains an irrational number, say λ , then it contains all numbers of the form $p + q\lambda$ and is therefore everywhere-dense. In this case we shall call P an *irrational power set*. This definition is clearly independent of the choice of base function.

Here again the set is either *related* (with a canonical form) or *semi-related*, an example of a semi-related irrational power set being the set $[f, g]$ of functions considered in C.F. II.

Two statements which are probably true and which we hope to prove later are as follows:

If the set of indices of a power set P is non-enumerable, then P is

* For example, Λ may consist of all the rationals, or of all the numbers of the form $p/2^n$.

a related set, i.e. all its members can be expressed in canonical form simultaneously.

Given any enumerable everywhere-dense set Λ , then there is a semi-related power set for which Λ is the set of indices.

8. Conclusion

The ideas discussed in the present paper are partly algebraic and partly functional, the properties of power sets described in §§ 5, 6 being largely algebraic while the definition of a power set is primarily functional, as also are certain aspects of the classification in § 7. There remain several problems, of a functional nature, arising out of the existence of certain semi-related classes.

Recalling our statement of the definition of a power set, we observe that although all three parts describe necessary properties, it is not certain that the definition is expressed as economically as possible. It is not clear, for example, that (ii) is independent of (iii), nor is it certain that (iii) is a minimum statement. In connexion with this latter point it may be mentioned that the algebraic properties of commutative functions are sometimes misleading, for it appears at first sight that (iii) can be replaced by the following:

(iii)' *There is a member of P , say f , which is node-free in (a, b) and such that if g is any other member and p, q any integers, then $f^p g^q$ do not intersect in (a, b) unless they are identical.*

From (i), (ii), and (iii)' it can be deduced that each member of P has an index with respect to f as in § 5, and that P can be augmented by the addition of all functions of the form $f^p g^q$, the index of this function being $p + q\lambda$, where λ is the index of g . Also, if members g, h have indices λ, μ respectively, then the product gh can be added to P and has index $\lambda + \mu$ provided λ and μ are not irrational with a rational sum. This proviso can be shown to be necessary because of certain cases involving semi-related functions. It follows that with (iii)' in place of (iii), the index law is not always satisfied, and two different members of P can in consequence have the same index, contrary to the properties which we wish functional powers to possess. Thus (iii)' cannot logically replace (iii).

Other possible alternatives to (iii) have been examined and found inadequate. It is still possible, however, that there exists a more economical alternative with the same content, i.e. a more precise set of tests to be applied to a given set of functions.

ON A TAUBERIAN THEOREM OF K. ANANDA RAU

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1. Not long ago L. S. Bosanquet (1) gave a short and simple proof of a Tauberian theorem left incomplete by Ananda Rau and substantially completed by Minakshisundaram (5, Theorem 1). This note shows how Bosanquet's method can be employed to prove the essentials of Ananda Rau's theorem, and of its companion due to V. Ganapathy Iyer (4, Theorem 4 and 5, Theorem 2) in more general forms which connect them with a Tauberian theorem of L. J. Mordell on the $(C, 1)$ mean (7) recently extended by us to Riesz means of order $r > 0$ (6, Theorem 1). A further point emerging from the note is that Ananda Rau's theorem has, in all cases but one, a counterpart, Corollary B₃ below, in which his 'O' hypothesis is replaced by the corresponding 'o' hypothesis and vice versa.

2. Let $\sum_{n=0}^{\infty} a_n$ be a given series and $\{\lambda_n\}$ a sequence such that

$$0 < \lambda_0 < \lambda_1 < \dots < \lambda_n < \dots \quad \text{and} \quad \lambda_n \rightarrow \infty \quad (n \rightarrow \infty).$$

Let

$$A^r(\omega) = \sum_{\lambda_n \leq \omega} (\omega - \lambda_n)^r a_n = r \int_0^{\omega} (\omega - t)^{r-1} A(t) dt \quad (r > 0),$$

where $A(t) = A^0(t) = \sum_{\nu=0}^n a_{\nu} \quad (\lambda_n \leq t < \lambda_{n+1})$.

Then Ananda Rau's theorem in the more general form referred to above can be stated thus:

THEOREM A. *Suppose that*

(i) $\phi(x), \psi(x)$ are positive functions of positive x such that

$$\theta(x) = \{\phi(x)/\psi(x)\}^{1/(r+1)} \leq x \quad (r > 0); \quad (2.1)$$

(ii) $\phi(x)$ is non-decreasing, and there are constants $H \geq 1, \eta > 0$ such that

$$\phi(x')/\phi(x) \leq H, \quad (2.2)$$

$$\psi(x')/\psi(x) \leq H, \quad (2.3)$$

when $0 < x' - x \leq \eta\theta(x)$;

* In view of (2.1) we can, of course, replace $0 < x' - x \leq \eta\theta(x)$ by $0 < x' - x \leq \eta x$.

$$(iii) \quad |a_n|/(\lambda_n - \lambda_{n-1}) \leq K\psi(\lambda_n), \quad (2.4)$$

$$(iv) \quad |A^r(\omega)| \leq \delta\phi(\omega), \quad (2.5)$$

where K, δ are positive constants;

$$\text{then} \quad A(\omega) = O[\vartheta(\omega)] \quad (\omega \rightarrow \infty). \quad (2.6)$$

where ϑ is defined as the function $\vartheta(x) = \phi(x)/\{\theta(x)\}^r$.

2.1. The following lemmas are required in the proof of Theorem A.

LEMMA 1. If $0 \leq \xi \leq \omega$ and p denotes the greatest integer less than $r > 0$, then

$$\frac{\Gamma(r+1)}{\Gamma(r-p)\Gamma(p+1)} \left| \int_0^\xi (\omega-t)^{r-p-1} A^p(t) dt \right| \leq \max_{0 \leq t \leq \xi} |A^r(t)|.$$

This is Lemma 8 of Hardy and Riesz (2).

LEMMA 2. Under the hypotheses (2.1), (2.3), (2.4),

$$|A(t) - A(\omega)| \leq KH\epsilon\vartheta(\omega), \quad (2.11)$$

provided that

$$\lambda_m \leq \omega < t \leq \lambda_m + \epsilon\theta(\omega) \quad (0 < \epsilon \leq \eta)$$

and (as we may suppose without loss of generality) that $\omega < \lambda_{m+1}$.

Proof. Let

$$\lambda_q \leq \lambda_m + \epsilon\theta(\omega) < \lambda_{q+1} \quad (m \leq q), \quad (2.12)$$

$$\lambda_\nu \leq t < \lambda_{\nu+1} \quad (m \leq \nu \leq q).$$

Then, if $m+1 \leq \nu \leq q$, using successively (2.4), (2.3), (2.12), we find that

$$\begin{aligned} |A(t) - A(\omega)| &\leq |a_{m+1}| + |a_{m+2}| + \dots + |a_\nu| \\ &\leq K[(\lambda_{m+1} - \lambda_m)\psi(\lambda_{m+1}) + \dots + (\lambda_\nu - \lambda_{\nu-1})\psi(\lambda_\nu)] \\ &\leq KH(\lambda_\nu - \lambda_m)\psi(\omega) \leq KH(\lambda_q - \lambda_m)\psi(\omega) \\ &\leq KH\epsilon\theta(\omega)\psi(\omega) = KH\epsilon\vartheta(\omega). \end{aligned}$$

Since $A(t) - A(\omega) = 0$ if $\nu = m$, the proof is complete.

The next two lemmas can be easily proved by induction.

LEMMA 3. For any function $F(x)$ of x and $\varphi > 0$, let us write

$$\Delta_\varphi^n F(x) = \sum_{\nu=0}^n (-1)^\nu \binom{n}{\nu} F(x + \overline{n-\nu}\varphi) \quad (n = 1, 2, \dots).$$

Then

$$\int_x^{x+\varphi} F(x+\varphi-t) \Delta_\varphi^\nu G(t) dt = \Delta_\varphi^\nu \int_x^{x+\varphi} F(x+\varphi-t) G(t) dt,$$

where $F(t), G(t)$ are integrable for all t in question.

LEMMA 4.

$$\Delta_{\varphi}^p A^p(t_0) = \Gamma(p+1) \int_{t_0}^{t_0+\varphi} dt_1 \int_{t_1}^{t_1+\varphi} dt_2 \int_{t_2}^{t_2+\varphi} \dots \int_{t_{p-1}}^{t_{p-1}+\varphi} A(t) dt \quad (p=1, 2, \dots).$$

2.2. *Proof of Theorem A.* If $r > 1$, let p be the greatest integer less than r . Then Lemma 4 is equivalent to

$$\varphi^p A(\omega) = \frac{\Delta_{\varphi}^p A^p(x)}{\Gamma(p+1)} + \int_x^{x+\varphi} dt_1 \int_{t_1}^{t_1+\varphi} dt_2 \int_{t_2}^{t_2+\varphi} \dots \int_{t_{p-1}}^{t_{p-1}+\varphi} \{A(\omega) - A(t)\} dt.$$

In this relation, let us take $\lambda_m \leq \omega < \lambda_{m+1}$, $(p+1)\varphi = \epsilon\theta(\omega)$; multiply both sides by $(\lambda_m + \varphi - x)^{r-p-1}$, and then integrate with respect to x from λ_m to $\lambda_m + \varphi$. The result is

$$\begin{aligned} \frac{\varphi^r A(\omega)}{r-p} &= \frac{1}{\Gamma(p+1)} \int_{\lambda_m}^{\lambda_m+\varphi} (\lambda_m + \varphi - x)^{r-p-1} \Delta_{\varphi}^p A^p(x) dx + \\ &+ \int_{\lambda_m}^{\lambda_m+\varphi} (\lambda_m + \varphi - x)^{r-p-1} dx \int_x^{x+\varphi} dt_1 \int_{t_1}^{t_1+\varphi} \dots \int_{t_{p-1}}^{t_{p-1}+\varphi} \{A(\omega) - A(t)\} dt \\ &= I + J \quad \text{say.} \end{aligned} \quad (2.21)$$

In virtue of Lemma 3,

$$\begin{aligned} I &= \frac{\Delta_{\varphi}^p}{\Gamma(p+1)} \int_{\lambda_m}^{\lambda_m+\varphi} (\lambda_m + \varphi - x)^{r-p-1} A^p(x) dx \\ &= \frac{1}{\Gamma(p+1)} \sum_{\nu=0}^p (-1)^{\nu} \binom{p}{\nu} \int_{\lambda_m+\overline{p-\nu}\varphi}^{\lambda_m+\overline{p-\nu+1}\varphi} (\lambda_m + \overline{p-\nu+1}\varphi - x)^{r-p-1} A^p(x) dx. \end{aligned}$$

This, in conjunction with Lemma 1, (2.5), and (2.2), gives

$$\begin{aligned} |I| &\leq \frac{\Gamma(r-p)}{\Gamma(r+1)} 2\delta \sum_{\nu=0}^p \binom{p}{\nu} \phi(\lambda_m + \overline{p-\nu+1}\varphi) \\ &\leq \frac{\Gamma(r-p)}{\Gamma(r+1)} 2\delta \sum_{\nu=0}^p \binom{p}{\nu} \phi(\lambda_m + \overline{p+1}\varphi) \\ &\leq \frac{\Gamma(r-p)}{\Gamma(r+1)} 2^{p+1} H\delta \phi(\omega). \end{aligned} \quad (2.22)$$

By Lemma 2,

$$|J| \leq KH\epsilon \frac{\varphi^r}{r-p} \vartheta(\omega). \quad (2.23)$$

From (2.21), (2.22), (2.23) we now get

$$\frac{\varphi^r |A(\omega)|}{r-p} \leq 2^{p+1} \frac{\Gamma(r-p)}{\Gamma(r+1)} H \delta \phi(\omega) + KH\epsilon \frac{\varphi^r}{r-p} \vartheta(\omega),$$

whence, remembering that $\varphi = \epsilon \theta(\omega)/(p+1)$, we are led to (2.6) in the form

$$|A(\omega)| \leq \left\{ 2^{p+1}(p+1)^r \frac{\Gamma(r-p+1)}{\Gamma(r+1)} H \frac{\delta}{\epsilon^r} + KH\epsilon \right\} \vartheta(\omega). \quad (2.24)$$

If $0 < r \leq 1$, the identity (2.21) becomes more simply

$$\frac{\varphi^r}{r} A(\omega) = \int_{\lambda_m}^{\lambda_m + \varphi} (\lambda_m + \varphi - x)^{r-1} A(x) dx + \int_{\lambda_m}^{\lambda_m + \varphi} (\lambda_m + \varphi - x)^{r-1} \{A(\omega) - A(x)\} dx$$

and therefore (2.24) continues to be valid with $p = 0$.

2.3. If, in (2.5), δ can be made arbitrarily small for all sufficiently large ω , an argument used by us elsewhere (6, Corollary 1.1) shows that we can take $\delta = \epsilon^{r+1}$ when $\omega \geq \omega_0(\epsilon)$ in (2.24) and reach

COROLLARY A₁. In Theorem A, replacement of (2.5) by

$$A^r(\omega) = o[\phi(\omega)]$$

alters the conclusion (2.6) to $A(\omega) = o[\vartheta(\omega)]$.

COROLLARY A₂. In the preceding corollary, the special choice $\phi(x) = x^\beta$ and $\psi(x) = x^\alpha$, with $0 \leq \beta \leq \alpha + r + 1$ gives the essential part of Ananda Rau's theorem.*

Other suitable choices of L -functions, $x^\alpha (\log x)^{\alpha_1} (\log_2 x)^{\alpha_2} \dots (\log_p x)^{\alpha_p}$ for $\phi(x)$ and $\psi(x)$, lead to extensions of Ananda Rau's theorem.

2.4. Remarks on Theorem A. (i) When $\sum a_n$ is a real series, § 2.2 makes it plain that Theorem A can be restated with (2.4) and (2.6) replaced by $a_n/(\lambda_n - \lambda_{n-1}) \leq K\psi(\lambda_n)$ and $A(\omega) = O_L[\vartheta(\omega)]$ respectively; further, as in Corollary A₁, the alteration of (2.5) to $A^r(\omega) = o[\phi(\omega)]$ secures the result $A(\omega) = o_L[\vartheta(\omega)]$.

(ii) After the demonstration in § 2.2, it is easy to prove the result: If $\phi(x)$ is non-decreasing and if, corresponding to any $\eta > 0$, there is an $H \geq 1$ such that $\phi(x')/\phi(x) \leq H$ when $0 < x' - x \leq \eta x$, and if $\lim \lambda_{n+1}/\lambda_n > 1$ ($n \rightarrow \infty$), then $A^r(\omega) = O[\phi(\omega)]$ implies

$$A(\lambda_n) = O[\phi(\lambda_n)/\lambda_n^r].$$

* Following Bosanquet (1, 243-4) we can replace $\beta \geq 0$ by $\beta > -1$.

This result admits of an extension which can be proved in the same way: If $\phi(x)$ is non-decreasing, then $A^r(\omega) = O[\phi(\omega)]$ implies

$$A(\lambda_n) = O[\phi(\lambda_{n+1})/(\lambda_{n+1} - \lambda_n)^r].$$

These two results are valid with o in place of O and therefore include two well-known theorems of Hardy and Riesz (2, Theorems 36, 21). Incidentally we note that the use of a 'Vandermonde-Cauchy' determinant in the proofs of these theorems may be avoided.

3. A theorem related to Theorem A exactly as Ganapathy Iyer's theorem (4, Theorem 2) is related to that of Ananda Rau, and carrying with it corollaries analogous to those in § 2.3, is the following.

THEOREM A'. In the hypotheses of Theorem A let (2.1) and (2.4) be replaced respectively by (2.1') and (2.4'):

$$\theta(x) = \{\phi(x)/\psi(x)x^{1/k}\}^{1/(r+1/k)} \leq x, \quad (2.1')$$

where $k > 1$, $\frac{1}{k} + \frac{1}{k'} = 1$, $r > 0$;

$$\sum_{v=1}^n |a_v|^k \lambda_v^k (\lambda_v - \lambda_{v-1})^{1-k} \leq K \{\psi(\lambda_n)\}^k \lambda_n^{k+1}. \quad (2.4')$$

Then Theorem A remains true provided that in the rest of the enunciation $\theta(x)$ is understood to refer to its new representation in (2.1').

The proof of Theorem A suffices for Theorem A' if, instead of Lemma 2, we use

LEMMA 2'. If, in the hypotheses of Lemma 2, $\theta(x)$ assumes its new form in (2.1') and if (2.4) is replaced by (2.4'), then

$$|A(t) - A(\omega)| < C \epsilon^{1/k'} \vartheta(\omega), \quad (3.1)$$

where C is a constant.

Proof. The modification required in the proof of Lemma 2, apart from the alteration in $\theta(x)$ to be carried out in (2.12), is based on the fact that, when $m+1 \leq v \leq q$,

$$\begin{aligned} |A(t) - A(\omega)| &\leq |a_{m+1}| + |a_{m+2}| + \dots + |a_v| \\ &= \sum_{n=1}^{v-m} |a_{m+n}| \lambda_{m+n} (\lambda_{m+n} - \lambda_{m+n-1})^{(1-k)/k} \frac{(\lambda_{m+n} - \lambda_{m+n-1})^{1/k'}}{\lambda_{m+n}} \\ &\leq \left\{ \sum_{n=1}^{v-m} |a_{m+n}|^k \lambda_{m+n}^k (\lambda_{m+n} - \lambda_{m+n-1})^{1-k} \right\}^{1/k} \left\{ \sum_{n=1}^{v-m} \frac{\lambda_{m+n} - \lambda_{m+n-1}}{\lambda_{m+n}^{k'}} \right\}^{1/k'} \end{aligned}$$

by Hölder's inequality; whence, from (2.4'),

$$|A(t) - A(\omega)| \leq K\psi(\lambda_\nu)\lambda_\nu^{1+1/k} \frac{(\lambda_\nu - \lambda_m)^{1/k'}}{\lambda_{m+1}}. \quad (3.2)$$

Using (2.12) and (2.3) with the altered $\theta(x)$, we find that

$$\lambda_\nu - \lambda_m \leq \lambda_q - \lambda_m \leq \epsilon \theta(\omega), \quad (3.3)$$

$$\begin{aligned} \frac{\psi(\lambda_\nu)\lambda_\nu^{1+1/k}}{\lambda_{m+1}} &< \frac{\psi(\lambda_\nu)\lambda_\nu^{1+1/k}}{\omega} \leq \frac{H\psi(\omega)\{\omega + \epsilon\theta(\omega)\}^{1+1/k}}{\omega} \\ &\leq \frac{H\psi(\omega)\{(1+\epsilon)\omega\}^{1+1/k}}{\omega}. \end{aligned} \quad (3.4)$$

Then taking together (3.2), (3.3), (3.4), we conclude that

$$|A(t) - A(\omega)| < C\epsilon^{1/k'}\psi(\omega)\omega^{1/k}\{\theta(\omega)\}^{1/k'},$$

which is (3.1) when $\nu \geq m$. Since (3.1) is trivial when $\nu = m$, the lemma is completely proved.

Thus (2.4') is virtually the extension of (2.4) in the sense that the effect of the latter can be secured by putting $1/k = 0$, $1/k' = 1$ in the former.

4. In certain cases it is useful to particularize the constant multiplying $\vartheta(\omega)$ in a manner different from that of (2.24). This is done in the next theorem.

THEOREM B. *In Theorem A let the hypotheses (i), (iii), (iv) remain unaltered, and replace (ii) by the hypothesis:*

$\phi(x)$ is non-decreasing and, corresponding to any $\eta > 0$, there is a constant $H \geq 1$ such that

$$\phi(x')/\phi(x) \leq H \text{ when } 0 < x' - x \leq \eta x, \quad (4.1)$$

and $\psi(x)$ satisfies the same condition as $\phi(x)$ in (4.1). (4.2)

$$\text{Then} \quad |A(\omega)| \leq CK^{r(1+r)}\delta^{1/(1+r)}\vartheta(\omega), \quad (4.3)$$

where C is a constant and $\vartheta(\omega)$ is defined as in (2.6).

4.1. To establish this theorem we require, in place of Lemma 2, the following rather obvious relations.

LEMMA 5. *With the hypothesis (2.4) and the assumption that $\psi(x)$ is non-decreasing, we have*

$$|A(\omega)| \leq K\omega\psi(\omega), \quad \lambda_m \leq \omega < \lambda_{m+1}. \quad (4.11)$$

LEMMA 6. *Under the conditions imposed on $\psi(x)$ in Theorem B,*

$$|A(t) - A(\omega)| \leq KH\psi(\omega)(t - \lambda_m), \quad (4.12)$$

provided that $\lambda_m \leq \omega < t \leq (1+\eta)\omega$ and (as we may suppose) $\omega < \lambda_{m+1}$.

4.2. *Proof of Theorem B.* First suppose that $r > 1$ and that p is the greatest integer less than r . In the identity (2.21) let $(p+1)\varphi \leq \eta\omega$, with the reservation that η, φ are to be fixed later subject to this condition. Then (2.22) is still true; further, in J ,

$$t \leq x + p\varphi \leq \lambda_m + (p+1)\varphi \leq (1+\eta)\omega,$$

so that (4.12) gives

$$|A(t) - A(\omega)| \leq KH\psi(\omega)(x + p\varphi - \lambda_m),$$

whence

$$|J| \leq \int_{\lambda_m}^{\lambda_{m+\varphi}} \varphi^p KH\psi(\omega)(x + p\varphi - \lambda_m)(\lambda_m + \varphi - x)^{r-p-1} dx.$$

Integrating by parts on the right, we get

$$|J| \leq KH\psi(\omega) \frac{\varphi^{r+1}}{r-p} \left(p + \frac{1}{r-p+1} \right). \quad (4.21)$$

Let us now fix

$$\eta = \frac{p+1}{\{p(r-p+1)+1\}},$$

$$\varphi = \frac{|A(\omega)|}{KH\psi(\omega)\{p(r-p+1)+1\}},$$

so that, on account of (4.11), $(p+1)\varphi \leq \eta\omega$. Then (4.21) gives

$$|J| \leq \frac{|A(\omega)|^{r+1}}{\{KH\psi(\omega)\}^r \{p(r-p+1)+1\}^r (r-p)(r-p+1)}. \quad (4.22)$$

In (2.21) we can use (4.22) and (2.22), which continues to hold, with the result that

$$\begin{aligned} \frac{\varphi^r |A(\omega)|}{r-p} &\leq \frac{\Gamma(r-p)}{\Gamma(r+1)} 2^{p+1} H \delta \phi(\omega) + \\ &+ \frac{|A(\omega)|^{r+1}}{\{KH\psi(\omega)\}^r \{p(r-p+1)+1\}^r (r-p)(r-p+1)}; \end{aligned}$$

i.e. in consequence of our definition of φ in terms of $|A(\omega)|$,

$$\frac{|A(\omega)|^{r+1}}{\{KH\psi(\omega)\}^r \{p(r-p+1)+1\}^r (r-p+1)} \leq \frac{\Gamma(r-p)}{\Gamma(r+1)} 2^{p+1} H \delta \phi(\omega),$$

which is equivalent to (4.3), C being defined by

$$C^{r+1} = \frac{\Gamma(r-p)}{\Gamma(r+1)} (r-p+1) \{p(r-p+1)+1\}^r 2^{p+1} H^{r+1}.$$

As explained in the proof of Theorem A, (4.3) with the above expression for C continues to be valid when $0 < r \leq 1$ provided that we put $p = 0$.

4.3. *Remarks on Theorem B.* (i) It is apparent that Theorem B is still true if, instead of (4.2), we suppose that $\psi(x)$ is non-increasing and that

$$\int_0^\omega \psi(x) dx = O[\omega\psi(\omega)].$$

(ii) The hypothesis (2.1) makes Theorem B and its modification in (i) non-trivial since otherwise, i.e. with $x < \theta(x)$, we can ensure $A(\omega) = O[\vartheta(\omega)]$ even without (4.1) and (2.5).

4.4. We conclude with some deductions from Theorem B.

COROLLARY B₁. *If, in Theorem B, K can be made as small as we please for all large n , the conclusion (4.3) will become $A(\omega) = o[\vartheta(\omega)]$; further, this result remains true when $\psi(x)$ is defined, not as in (4.2), but as in § 4.3 (i).*

In particular we have

COROLLARY B₂. *If*

(i) $a_n = o\{\lambda_n^\alpha(\lambda_n - \lambda_{n-1})\}$ where $\alpha + 1 > 0$,

(ii) $A^r(\omega) = O(\omega^\beta)$ where $r > 0$, $0 \leq \beta < \alpha + r + 1$,

then $A(\omega) = o(\omega^{(\alpha r + \beta)/(r+1)})$.

When $\alpha + 1 > 0$, $\beta \geq \alpha + r + 1$, the conclusion is true but trivial.

An argument of Bosanquet's (1, 243-4) shows that we can relax the restrictions on α , β and establish Corollary B₂ for $\beta > -1$ and any real $\alpha \neq -1$ simultaneously with $\beta = r$, in the form:

COROLLARY B₃. *If*

(i) $a_n = o\{\lambda_n^\alpha(\lambda_n - \lambda_{n-1})\}$ where α is real,

(ii) $A^r(\omega) = O(\omega^\beta)$ where $r > 0$ and $-1 < \beta < \alpha + r + 1$,

then

$$A^r(\omega) = o(\omega^{(\alpha(r-\gamma) + \beta(\gamma+1))/(r+1)})$$

when $0 \leq \gamma < r$; the conclusion being true but trivial when either $\alpha \neq -1$, $\beta \geq \alpha + r + 1$, or $\alpha = -1$, $\beta > r$.*

* It is evident that so long as $\alpha \geq 0$, $\beta \geq 0$, Theorem B includes the non-trivial contents of both Ananda Rau's theorem and its counterpart: the former when $\delta = \epsilon^{r+1}$, $\omega \geq \omega_0(\epsilon)$, and the latter when $K = \epsilon^{(r+1)/r}$, $n \geq n_0(\epsilon)$. Supplementing Theorem B with Bosanquet's argument, we can replace $\alpha \geq 0$, $\beta \geq 0$ by α real, $\beta > -1$.

That this corollary is false in the excluded case $\alpha = -1$, $\beta = r$ is seen from the following consideration. When $\alpha = -1$, $\beta = r = 1$, the hypotheses (i) and (ii) of Corollary B_3 employed in

$$A(\lambda_m) = \frac{A^1(\lambda_m)}{\lambda_m} + \frac{\sum_{n=1}^m a_n \lambda_n}{\lambda_m}$$

show that, as $m \rightarrow \infty$,

$$A(\lambda_m) = O(1) + o(1) \neq o(1),$$

which means that the conclusion is not true when $\gamma = 0$.

The particular choice $\beta = -\alpha r$, $-1 \neq \alpha < 1/r$ in Corollary B_3 , yields a theorem given in substance by Hyslop (3, Theorem 19) for $\lambda_n = n$, $r =$ a positive integer, $-1 < \alpha \leq (1-r)/r$, where in case the equality sign prevails $r > 1$.

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ON A FACTORIZATION OF PSEUDO-ORTHOGONAL MATRICES

By P. L. HSU (*Peiping*)

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A PSEUDO-ORTHOGONAL matrix is defined to be a matrix† A which satisfies

$$A\{I_p \dot{+} (-I_q)\}A' = I_p \dot{+} (-I_q), \quad (1)$$

where the accent denotes the transpose, $\dot{+}$ the direct sum, and I_p denotes the p -rowed unit matrix. For definiteness we assume that $p \leq q$. A factorization of A into a product of $\frac{1}{2}(p+q)(p+q-1)-p$ plane rotations and p pseudo-plane rotations has recently been worked out by H. C. Lee.‡ In this note I give a more explicit factorization which is unique in general.

LEMMA 1. Let U be any $p \times q$ matrix of rank r and let the latent roots of UU' be $\alpha_1, \dots, \alpha_p$, where $\alpha_i > 0$ ($i \leq r$), $\alpha_i = 0$ ($i > r$). Then there is a $p \times p$ orthogonal matrix Ω_1 and a $q \times q$ orthogonal matrix Ω_2 such that

$$U = \Omega_1[\alpha_1^{\frac{1}{2}} \dot{+} \dots \dot{+} \alpha_r^{\frac{1}{2}}, 0]\Omega_2.$$

Proof. Choose Ω_1 so that

$$\Omega_1 UU' \Omega_1' = \alpha_1 \dot{+} \alpha_2 \dot{+} \dots \dot{+} \alpha_r \dot{+} 0. \quad (2)$$

Let S be the matrix formed by the first r rows, and T that formed by the last $p-r$ rows, of Ω_1 . Then (2) gives

$$SUU'S' = \alpha_1 \dot{+} \dots \dot{+} \alpha_r, \quad TUU'T' = 0,$$

so that

$$TU = 0.$$

Hence the $r \times q$ matrix $G = (\alpha_1^{-\frac{1}{2}} \dot{+} \dots \dot{+} \alpha_r^{-\frac{1}{2}})SU$ has mutually orthogonal rows, i.e. $GG' = I_r$. Taking Ω_2 to be an orthogonal matrix whose first r rows are those of G we have

$$\begin{aligned} U &= (I - T'T)U = S'SU = S'(\alpha_1^{\frac{1}{2}} \dot{+} \dots \dot{+} \alpha_r^{\frac{1}{2}})G \\ &= \Omega_1' \begin{bmatrix} \alpha_1^{\frac{1}{2}} \dot{+} \dots \dot{+} \alpha_r^{\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix} \Omega_2 = \Omega_1'[\alpha_1^{\frac{1}{2}} \dot{+} \dots \dot{+} \alpha_p^{\frac{1}{2}}, 0]\Omega_2. \end{aligned}$$

† All the quantities in this paper are real.

‡ *Quart. J. of Math.* (Oxford), 15 (1944), 7-10. Owing to my inaccessibility to literature Lee's paper is the only one citable in this connexion. For the same reason lemmas, even when embodying known results, are given with proof.

LEMMA 2. If B is any $m \times m$ matrix, then there is an $m \times m$ orthogonal matrix Ω such that $B\Omega$ is of the triangular type, viz. a matrix whose super-diagonal elements are zero and whose diagonal elements are non-negative.

The proof may be found in Lee's paper.[†] The elements (1, 2), (1, 3), ..., (1, m), (2, 3), etc., are annihilated successively by means of appropriate orthogonal matrices representing plane rotations.

Having established the two lemmas, consider a pseudo-orthogonal matrix

$$A = \begin{bmatrix} B & C \\ D & E \end{bmatrix} \quad (3)$$

satisfying (1), where B is $p \times p$ and E is $q \times q$. Using (1) we have

$$(i) \quad BB' = I_p + CC', \quad (ii) \quad EE' = I_q + DD', \quad (iii) \quad BD' = CE'.$$

By (i) and (ii) B and E are non-singular. Hence, by (iii),

$$C = BU, \quad D = EU', \quad (4)$$

where U is some $p \times q$ matrix. Substituting in (i) and (ii) we get[‡]

$$(B'B)^{-1} = I_p - UU', \quad (E'E)^{-1} = I_q - U'U. \quad (5)$$

The latent roots of UU' being necessarily smaller than 1 in virtue of (5), let them be $\lambda_i/(1+\lambda_i)$ ($i = 1, \dots, p$). Following Lemma 1 we can write

$$U = \Omega_1 \left[\left(\frac{\lambda_1}{1+\lambda_1} \right)^{\frac{1}{2}} \dot{+} \dots \dot{+} \left(\frac{\lambda_p}{1+\lambda_p} \right)^{\frac{1}{2}} \dot{+} 0 \right] \Omega_2, \quad (6)$$

whence, by (5),

$$B'B = \Omega_1 \{ (1+\lambda_1) \dot{+} \dots \dot{+} (1+\lambda_p) \} \Omega_1, \\ E'E = \Omega_2 \{ (1+\lambda_1) \dot{+} \dots \dot{+} (1+\lambda_p) \dot{+} I_{q-p} \} \Omega_2.$$

Hence

$$B = \Gamma_1 \{ (1+\lambda_1) \dot{+} \dots \dot{+} (1+\lambda_p) \} \Omega_1, \\ E = \Gamma_2 \{ (1+\lambda_1) \dot{+} \dots \dot{+} (1+\lambda_p) \dot{+} I_{q-p} \} \Omega_2, \quad (7)$$

where Γ_1 is a $p \times p$ orthogonal matrix and Γ_2 is a $q \times q$ orthogonal matrix.

Taking (3), (7), (4), and (6) together we get

$$A = (\Gamma_1 \dot{+} \Gamma_2) (\Lambda \dot{+} I_{q-p}) (\Omega_1 \dot{+} \Omega_2), \quad (8)$$

where

$$\Lambda = \begin{bmatrix} (1+\lambda_1) \dot{+} \dots \dot{+} (1+\lambda_p) \dot{+} & \lambda_1^{\frac{1}{2}} \dot{+} \dots \dot{+} \lambda_p^{\frac{1}{2}} \\ \lambda_1^{\frac{1}{2}} \dot{+} \dots \dot{+} \lambda_p^{\frac{1}{2}} & (1+\lambda_1) \dot{+} \dots \dot{+} (1+\lambda_p) \dot{+} \end{bmatrix}.$$

[†] Ibid., p. 8.

[‡] Routine steps, such as pre- and post-multiplication by appropriate matrices, are left to the reader.

Since each of Γ_1 and Ω_1 depends on $\frac{1}{2}p(p-1)$ independent parameters, each of Γ_2 and Ω_2 depends on $\frac{1}{2}q(q-1)$ independent parameters, and Λ depends on p independent parameters, apparently the right-hand side of (8) depends on $p(p-1)+q(q-1)+p$ independent parameters, whilst A depends on $\frac{1}{2}(p+q)(p+q-1)$ independent parameters by its definition, the discrepancy being $\frac{1}{2}(q-p)(q-p-1)$. The factorization (8) cannot be unique.

If Δ is any $(q-p)$ -rowed orthogonal matrix, $I_{2p} \dot{+} \Delta'$ is commutative with the middle factor on the right of (8). Hence

$$\begin{aligned} A &= (\Gamma_1 \dot{+} \Gamma_2)(I_{2p} \dot{+} \Delta)(\Lambda \dot{+} I_{q-p})(I_{2p} \dot{+} \Delta')(\Omega_1 \dot{+} \Omega_2) \\ &= \{\Gamma_1 \dot{+} \Gamma_2(I_p \dot{+} \Delta)\}(\Lambda \dot{+} I_{q-p})\{\Omega_1 \dot{+} (I_p \dot{+} \Delta')\Omega_2\}. \end{aligned} \quad (9)$$

The last factor in (9) is of the same nature as that in (8), so it can still be written as $\Omega_1 \dot{+} \Omega_2$, with a changed Ω_2 .

Writing

$$\Gamma_2 = \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{bmatrix},$$

where Π_1 is $p \times p$ and Π_2 is $(q-p) \times (q-p)$, we have

$$\Gamma_2(I_p \dot{+} \Delta) = \begin{bmatrix} \Pi_1 & \Pi_2 \Delta \\ \Pi_3 & \Pi_4 \Delta \end{bmatrix}.$$

We select Δ , in accordance with Lemma 2, so that $\Pi_4 \Delta$ is of the triangular type. In this way the number of parameters in Γ_2 is reduced by $\frac{1}{2}(q-p)(q-p-1)$, which accounts for the discrepancy mentioned above. We summarize all the foregoing results in the following theorem.

THEOREM 1. *The following factorization exists for every pseudo-orthogonal matrix that leaves $I_p \dot{+} (-I_q)$ ($p \leq q$) invariant:*

$$A = (\Gamma_1 \dot{+} \Gamma_2)(\Lambda \dot{+} I_{q-p})(\Omega_1 \dot{+} \Omega_2), \quad \Gamma_2 = \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{bmatrix}, \quad (10)$$

where Γ_1 and Ω_1 are $p \times p$ orthogonal matrices, Γ_2 and Ω_2 are $q \times q$ orthogonal matrices, and Π_4 is of the triangular type.

We now examine the uniqueness of the factorization (10).

THEOREM 2. *The factorization (10) is unique under the following conditions:*

- (i) $\lambda_1, \dots, \lambda_p$ are all distinct and arranged in descending order;
- (ii) $\lambda_i > 0$ ($i = 1, \dots, p$);
- (iii) Π_4 is non-singular.

Proof. The middle factor of every factorization (10) is the same, because $1+\lambda_1, \dots, 1+\lambda_p$ are the latent roots of BB' , where B is the sub-matrix in (3), arranged in descending order. Let

$$(\Gamma_1^* \dot{+} \Gamma_2^*)(\Lambda \dot{+} I_{q-p})(\Omega_1^* \dot{+} \Omega_2^*), \quad \text{where} \quad \Gamma_2^* = \begin{bmatrix} \Pi_1^* & \Pi_2^* \\ \Pi_3^* & \Pi_4^* \end{bmatrix}, \quad (11)$$

be a second factorization of A . Then

$$(\Gamma_1^* \dot{+} \Gamma_2^*)(\Gamma_1 \dot{+} \Gamma_2)(\Lambda \dot{+} I_{q-p}) = (\Lambda \dot{+} I_{q-p})(\Omega_1^* \dot{+} \Omega_2^*)(\Omega_1 \dot{+} \Omega_2)'. \quad (12)$$

Denoting by

$$\begin{bmatrix} G & 0 & 0 \\ 0 & H_1 & H_2 \\ 0 & H_3 & H_4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} L & 0 & 0 \\ 0 & M_1 & M_2 \\ 0 & M_3 & M_4 \end{bmatrix}$$

respectively the pre-multiplier and the post-multiplier of $\Lambda \dot{+} I_{q-p}$ in (12) we obtain, on equating corresponding blocks,

$$\begin{aligned} GD_1 &= D_1 L, & GD_2 &= D_2 M_1, & 0 &= D_2 M_2, \\ H_1 D_2 &= D_2 L, & H_1 D_1 &= D_1 M_1, & H_2 &= D_1 M_2, \\ H_3 D_2 &= 0, & H_3 D_1 &= M_3, & H_4 &= M_4, \end{aligned}$$

where $D_1 = (1+\lambda_1)^{\frac{1}{2}} \dot{+} \dots \dot{+} (1+\lambda_p)^{\frac{1}{2}}$ and $D_2 = \lambda_1^{\frac{1}{2}} \dot{+} \dots \dot{+} \lambda_p^{\frac{1}{2}}$. By condition (ii), D_2 is non-singular, hence $H_2 = H_3 = M_2 = M_3 = 0$. From the first equation we get $GD_1^2 G' = D_1 LL' D_1 = D_1^2$, whence $GD_1^2 = D_1^2 G$, giving $G = I_p$ since the λ_i are all distinct. Then $L = H_1 = M_1 = I_p$. Hence

$$\Gamma_1 \dot{+} \Gamma_2 = (\Gamma_1^* \dot{+} \Gamma_2^*)(I_{2p} \dot{+} H_4).$$

In particular $\Pi_4 = \Pi_4^* H_4$. Since Π_4 is non-singular, so is Π_4^* and we have $H_4 = (\Pi_4^*)^{-1} \Pi_4$, a matrix of the triangular type. Hence $H_4 = I_{q-p}$ and so the three factors in (11) are identical with the corresponding factors in (10). This completes the proof.

As a final remark we add that Lee's factorization may be deduced from (10). For the matrix A is factorized into the factors

$$\Gamma_1 \dot{+} I_q, \quad I_p \dot{+} \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{bmatrix}, \quad \Lambda \dot{+} I_{q-p}, \quad \Omega_1 \dot{+} I_q, \quad I_p \dot{+} \Omega_2.$$

Clearly the first and the last matrices are each factorizable into $\frac{1}{2}p(p-1)$ plane rotations, the second into $\frac{1}{2}q(q-1) - \frac{1}{2}(q-p)(q-p-1)$ plane rotations, the fourth into $\frac{1}{2}q(q-1)$ plane rotations, and, finally, Λ is factorizable into p pseudo-plane rotations.

THE ARITHMETIC MINIMA OF POSITIVE QUADRATIC FORMS (I)

By T. W. CHAUNDY (*Oxford*)

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1. Introduction

THE general quadratic

$$F = \sum_1^n a_{rr} x_r^2 + 2 \sum_{r>s}^n a_{rs} x_r x_s$$

can be written, as we know, in a form

$$\alpha_1(x_1 + b_{12}x_2 + \dots + b_{1n}x_n)^2 + \dots + \alpha_{n-1}(x_{n-1} + b_{n-1,n}x_n)^2 + \alpha_n x_n^2. \quad (1)$$

Restrict the coefficients a_{rr} , a_{rs} to be real, the quadratic to be positive-definite, and the variables x to be positive or negative integers or zero, the case in which every x is zero being excluded. Then every α_r is positive, and the b_{rs} are real.

It then follows from (1) that F is less than a given positive m only for a finite number of (integer) sets of the variables x ; and therefore that F , restricted to integer variables, has an attained minimum, which we can think of as its 'arithmetic minimum', $\min F$.

1.1. If we now vary the coefficients of F , we can look for a possible maximum of this minimum. But a homogeneous change of coefficients from a_{rr} , a_{rs} to ta_{rr} , ta_{rs} (with arbitrary positive t) multiplies $\min F$ by t , and so $\min F$ itself is unbounded. We therefore compare $\min F$ with the determinant $\Delta = |a_{rs}|$ of F ; or, more precisely, taking account of dimensions in the coefficients, with $\sqrt[n]{\Delta}$, looking for a possible maximum γ_n of

$$\frac{\min F}{\sqrt[n]{\Delta}}.$$

Such a maximum γ_n , if it exists, is a number, independent of the coefficients and depending only on n , for which integers x_1, \dots, x_n not all zero exist such that

$$F \leq \gamma_n \sqrt[n]{\Delta};$$

and γ_n will be a 'best possible' value, in the sense that the inequality would not hold for every n -ary quadratic F if γ_n were replaced by a smaller number. A quadratic F for which equality is attained, i.e. such that

$$\min F = \gamma_n \sqrt[n]{\Delta},$$

is known as an *extreme* form.

Any (homogeneous) linear substitution of unit determinant on the variables leaves Δ unaltered. If the coefficients of the substitution are integers, it leaves unaffected the integer character of the variables. Moreover, it transforms the 'origin', i.e. all $x = 0$, into itself. The arithmetic minima are therefore unaffected by unimodular, integer substitutions. Two quadratics so derived from one another are regarded as 'equivalent' and, for our present purposes, need not be separately considered.

The above is elementary and well known, but it has seemed useful to provide a brief introduction for those who (like myself recently) need to begin from the beginning.

1.2. The problem for general n seems first to have been considered by Hermite (3) in 1850, and the following values of γ_n have been obtained:

$$\begin{array}{llll} \gamma_1 = 1, & \gamma_2 = \sqrt{\frac{4}{3}}, & \gamma_3 = \sqrt[3]{2}, & \gamma_4 = \sqrt[4]{4}, \\ \gamma_5 = \sqrt[5]{8}, & \gamma_6 = \sqrt[6]{\frac{64}{3}}, & \gamma_7 = \sqrt[7]{64}, & \gamma_8 = 2. \end{array}$$

Of these γ_2 and γ_3 are due to Lagrange and to Gauss (2), γ_3 to γ_5 were given by Korkine and Zolotareff (5), (6). More recently γ_6 was found by Hofreiter (4) and γ_6 to γ_8 by Blichfeldt (1), who gives references to other literature. Quite recently Mordell (7) has shown that

$$(\gamma_n)^{n-2} \leq (\gamma_{n-1})^{n-1}, \quad (2)$$

with equality at $n = 4, 8$ as is seen from the table. In §§ 9, 10 of this paper I prove that $\gamma_9 = 2$ and $\gamma_{10} = 2 \sqrt[10]{\frac{4}{3}}$.

2. An inductive method

I develop here an inductive method which, so far as it goes, is considerably simpler than that generally used. In the definition of γ_n as the maximum of the ratio $(\min F)^{1/n}/\Delta$, it is often usual to fix Δ as unity, so that γ_n is just the maximum of $\min F$ itself under this restriction. Here I prefer to fix $\min F$ at the convenient value 2 and to look for Δ_n , the least value of the corresponding Δ . We have at once

$$\Delta_n = (2/\gamma_n)^n,$$

with the corresponding table of known Δ_n :

$$\begin{array}{llll} \Delta_1 = 2, & \Delta_2 = 3, & \Delta_3 = 4, & \Delta_4 = 4, \\ \Delta_5 = 4, & \Delta_6 = 3, & \Delta_7 = 2, & \Delta_8 = 1. \end{array}$$

Mordell's inequality (2) becomes

$$\Delta_n^{(n-2)/n} \geq \frac{1}{2} \Delta_{n-1}. \quad (3)$$

2.1. Thus always here, except when all the x are zero,

$$F(x_1, \dots, x_n) \geq 2, \quad (4)$$

where (I emphasize) the minimum is actually attained at some point (x_1, \dots, x_n) .^{*} Since

$$F(px_1, \dots, px_n) = p^2 F(x_1, \dots, x_n) > F(x_1, \dots, x_n)$$

if p is an integer exceeding unity, the coordinates of such a minimum point are co-prime. I note the lemma:

LEMMA 1. *Any given point (x_1, \dots, x_n) whose coordinates are co-prime and not all zero can be transformed by a unimodular, integer linear substitution into the unit point $(1, 0, \dots, 0)$.*

We can use this lemma (replacing F , if need be, by an equivalent F) to secure that F attains its minimum at the unit point $(1, 0, \dots, 0)$ and hence that always

$$a_{11} = 2,$$

and so, trivially, $\Delta_1 = a_{11} = 2$.

2.2. The method proceeds by considering the behaviour of the restricted minima of F when some only of the x are allowed to vary: at first x_1 by itself, then x_1 and x_2 , then x_1, x_2, x_3 , and so on. In this inductive fashion we shall be able to build up successively the values of $\Delta_2, \Delta_3, \Delta_4, \dots$ with their corresponding extreme forms. This involves going over some familiar ground and rediscovering a number of known results, but the details of the method are best revealed, I think, by this progressive development of the argument through increasing values of n .

2.3. I begin, then, by supposing all the variables except (say) x_1 fixed at arbitrary, unspecified values, which, of course, may be the coordinates of a minimum point: simultaneous zeros are alone excepted. Write F in the form

$$F = 2X^2 + G_2(x_2, \dots, x_n), \quad (5)$$

* The notation ' $f(x) \geq h$ ' can be ambiguous in that it may imply (i) that equality is actually attained for some x , or (ii) that we merely know that $f(x)$ is never less than h . Here I shall always use the symbol in the first sense, writing ' $f(x) \leq h$ ' for the vaguer (ii).

where

$$X = x_1 + \frac{1}{2}a_{12}x_2 + \frac{1}{2}a_{13}x_3 + \dots + \frac{1}{2}a_{1n}x_n,$$

$$G_2 = c_{22}x_2^2 + 2c_{23}x_2x_3 + \dots, \text{ say.}$$

By the ordinary theory of quadratic forms the determinants of F , G_2 are connected by the relation

$$\Delta(F) = 2\Delta(G_2). \quad (6)$$

Now $F \geq 2$ with equality at $(1, 0, \dots, 0)$ but not necessarily elsewhere. Thus, when x_1, \dots, x_n are not all zero, we have from (5)

$$G_2(x_2, \dots, x_n) \leq 2(1 - X^2). \quad (7)$$

We can write

$$\frac{1}{2}a_{12}x_2 + \dots + \frac{1}{2}a_{1n}x_n = p \pm \theta, \quad (8)$$

where p is an integer or zero and $0 \leq \theta \leq \frac{1}{2}$.^{*} Then, for variation of x_1 , if x_2, \dots, x_n are not all zero, $\min X^2$ is θ^2 given by $x_1 = -p$. Thus, from (7),

$$G_2 \leq 2(1 - \theta^2), \quad \text{where } \theta^2 \leq \frac{1}{4}, \quad (9)$$

and we cannot improve on this inequality. We then get least $\Delta(G_2)$ if we take $\theta = \frac{1}{2}$ and write in (9)

$$G_2(x_2, \dots, x_n) \geq \frac{3}{2}. \dagger \quad (10)$$

For, if we put $G_2 \geq \frac{3}{2}t$ where $t > 1$, the homogeneous change of coefficients

$$c'_{rr}, c'_{rs} = c_{rr}/t, c_{rs}/t$$

would give a new $G'_2 (= G_2/t)$ in which

$$\Delta'(G_2) = \Delta(G_2)/t^{n-1} < \Delta(G_2), \quad \text{while still } G'_2 \geq \frac{3}{2}.$$

Thus, as in (10), we fix $\frac{3}{2}$ as the (attained) minimum of G_2 .

2.4. In particular, when $n = 2$ and $x_2 \neq 0$, we have $G_2 = c_{22}x_2^2$, so that

$$\Delta(G_2) = c_{22} \quad \text{and} \quad c_{22}x_2^2 \geq \frac{3}{2} \quad (x_2 \neq 0).$$

The minimum is at $x_2 = \pm 1$, and so, for least $\Delta(G_2)$, $c_{22} = \frac{3}{2}$. Thus, by (6),

$$\Delta_2 = \min \Delta(F_2) = 3. \quad (11)$$

^{*} Of course, when $\theta = \frac{1}{2}$, the two expressions $p + \theta$, $(p + 1) - \theta$ are equal. To avoid redundancy I shall use always the former. So, more generally, when a number of expressions $p_r + \theta_r$ are concerned, I use $p_r + \frac{1}{2}$ but not $p_r - \frac{1}{2}$.

[†] By (8), if x_2, \dots, x_n are not all zero, we can always choose a_{12}, \dots, a_{1n} so that $\theta = \frac{1}{2}$.

For the corresponding F_2 we have, from (8),

$$\frac{1}{2}a_{12} = p + \frac{1}{2},$$

$$\text{i.e.} \quad F_2 = 2\{x_1 + (p + \frac{1}{2})x_2\}^2 + \frac{3}{2}x_2^2.*$$

The unimodular integer substitution

$$x_1 + px_2 \rightarrow x_1, \quad x_2 \rightarrow x_2 \quad (12)$$

gives the simpler equivalent form

$$\Phi_2 = 2(x_1 + \frac{1}{2}x_2)^2 + \frac{3}{2}x_2^2 = 2(x_1^2 + x_1x_2 + x_2^2). \quad (13)$$

This then is the sufficiently general extreme form when $n = 2$, as is well known.

3. Variation of x_1, x_2

When $n > 2$ and x_2, \dots, x_n are not all zero, the minimum $\frac{3}{2}$ of G_2 will be attained at some point (x_2, \dots, x_n) with co-prime coordinates. By Lemma 1 a suitable unimodular integer substitution on (x_2, \dots, x_n) will convert this minimum point into the unit point $x_2 = 1, x_3 = 0, \dots, x_n = 0$. Combined with $x_1 \rightarrow x_1$ the substitution is unimodular and integer in the full n variables and therefore converts F into an equivalent F . Thus sufficiently $G_2 = \frac{3}{2}$ at $x_2 = 1, x_3 = 0, \dots, x_n = 0$, and so, as before,

$$c_{22} = \frac{3}{2}, \quad \frac{1}{2}a_{12} = p + \frac{1}{2},$$

where again sufficiently $p = 0$, since we can now use the substitution

$$x_1 + px_2 \rightarrow x_1, \quad x_r \rightarrow x_r \quad (r = 2, \dots, n)$$

in place of (12).

Thus the extreme forms are to be found only among

$$F = 2(x_1 + \frac{1}{2}x_2)^2 + \frac{3}{2}x_2^2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3 + a_{33}x_3^2 + \dots \quad (14)$$

or forms equivalent to these. That is to say, F consists of Φ_2 together with terms in which x_3, \dots, x_n appear (with or without x_1, x_2).

3.1. I now vary x_1, x_2 keeping x_3, \dots, x_n constant. Collecting all the terms containing x_1 or x_2 we can write F in the form

$$\frac{1}{2}(2x_1 + x_2 + \delta_1)^2 + \frac{3}{2}(x_2 + \delta_2)^2 + G_3(x_3, \dots, x_n),$$

* It should be noted that the argument secures the inequality $F_2 \geq 2$ only when $|x_2| = 1$, where the special values of p, θ are determined. Conceivably, when $|x_2| \geq 2$, these values might give to $2X^2$ a smaller minimum than $\frac{1}{2}$, and the condition $G_2 \geq \frac{3}{2}$ would then be insufficient to secure $F_2 \geq 2$. Fortunately in this doubtful case G_2 by itself exceeds 2, for $c_{22} > \frac{1}{2}$ and so $c_{22}x_2^2 > 2$ when $|x_2| \geq 2$. For the other values of n studied in this paper we find always that $c_{nn} \geq \frac{1}{2}$, and so the anticipated difficulty does not materialize.

where
$$\left. \begin{aligned} \delta_1 &= a_{13}x_3 + \dots + a_{1n}x_n \\ \frac{1}{2}\delta_1 + \frac{3}{2}\delta_2 &= a_{23}x_3 + \dots + a_{2n}x_n \end{aligned} \right\}. \quad (15)$$

We note that

$$\Delta(F) = \Delta_2 \Delta(G_3) = 3\Delta(G_3). \quad (16)$$

3.2. We next consider the minima of

$$\Theta_2 = \frac{1}{2}(2x_1 + x_2 + \delta_1)^2 + \frac{3}{2}(x_2 + \delta_2)^2$$

for integer or zero values of x_1, x_2 and fixed δ_1, δ_2 . As before we separate δ_1, δ_2 into integer and fractional parts by writing

$$\delta_1 = p_1 \pm \theta_1, \quad \delta_2 = p_2 \pm \theta_2 \quad (0 \leq \theta_1, \theta_2 \leq \frac{1}{2}),$$

so that
$$\Theta_2 = \frac{1}{2}(2x_1 + x_2 + p_1 \pm \theta_1)^2 + \frac{3}{2}(x_2 + p_2 \pm \theta_2)^2.$$

The two terms, considered independently, have the minima $\frac{1}{2}\theta_1^2, \frac{3}{2}\theta_2^2$, and these are attained simultaneously if the simultaneous equations

$$2x_1 + x_2 + p_1 = 0, \quad x_2 + p_2 = 0$$

have an integer solution. This requires $p_1 + p_2$ to be even. Thus, if x_3, \dots, x_n are not all zero, we can choose a_{13}, a_{23}, \dots in F so that $p_1 + p_2$ is odd, and then the minimum of Θ_2 is greater (or at least as great), and so, as in §2, $\Delta(G_3)$ and therefore $\Delta(F)$ are smaller (or at least as small).

With $p_1 + p_2$ odd, $2x_1 + x_2 + p_1, x_2 + p_2$ will be one odd and one even, so that $\min \Theta_2$ will be one of

$$\rho_1 = \frac{1}{2}\theta_1^2 + \frac{3}{2}(1 - \theta_2)^2, \quad \rho_2 = \frac{1}{2}(1 - \theta_1)^2 + \frac{3}{2}\theta_2^2;$$

precisely

$$\min \Theta_2 = \min(\rho_1, \rho_2).$$

3.3. It remains to determine the greatest value of this $\min \Theta_2$ for variation of θ_1, θ_2 . Now we can write

$$\frac{1}{2}(\rho_1 + \rho_2) = \frac{2}{3} - \frac{1}{2}\theta_1(1 - \theta_1) - \frac{3}{2}(\theta_2 - \frac{1}{3})(\frac{2}{3} - \theta_2),$$

$$\rho_2 = \frac{2}{3} - \frac{1}{2}\theta_1(2 - \theta_1) - \frac{3}{2}(\frac{1}{3} - \theta_2)(\frac{1}{3} + \theta_2).$$

Thus, since θ_1, θ_2 lie in the closed interval $(0, \frac{1}{2})$,

$$\frac{1}{2}(\rho_1 + \rho_2) < \frac{2}{3} \quad \text{if} \quad \theta_2 > \frac{1}{3};$$

$$\rho_2 < \frac{2}{3} \quad \text{if} \quad \theta_2 < \frac{1}{3}.$$

Hence $\min(\rho_1, \rho_2) < \frac{2}{3}$ unless $\theta_2 = \frac{1}{3}$. When $\theta_2 = \frac{1}{3}$, we still have $\min(\rho_1, \rho_2) < \frac{2}{3}$ unless $\theta_1 = 0$. Thus

$$\min \Theta_2 = \min(\rho_1, \rho_2) \leq \frac{2}{3}$$

with equality only when $\theta_1 = 0, \theta_2 = \frac{1}{3}$.

Thus, as arising from any F of the restricted form (14), $\Theta_2 \leq \frac{2}{3}$, with equality only when F is such that

$$p_1 + p_2 \text{ is odd, } \theta_1 = 0, \quad \theta_2 = \frac{1}{3}. \quad (17)$$

The condition $F = \Theta_2 + G_3 \geq 2$ then gives

$$G_3 \leq 2 - \frac{2}{3} = \frac{4}{3}.$$

As before we secure the least $\Delta(G)$ by taking

$$G_3(x_3, \dots, x_n) \geq \frac{4}{3},$$

where, as we have said, simultaneous zeros of all x_3, \dots, x_n are excluded.

3.4. When $n = 3$ and $x_3 \neq 0$, we have, as in § 2.4,

$$G_3 = c_{33} x_3^2 \geq \frac{4}{3}.$$

$$\Delta(G_3) = c_{33}, \quad \Delta(F_3) = \Delta_2 \Delta(G_3) = 3c_{33}.$$

The minimum of G_3 is at $x_3 = \pm 1$, and so $\min \Delta(G_3) = \frac{4}{3}$ and

$$\Delta_3 = \min \Delta(F_3) = 4. \quad (18)$$

When $n = 3$, F_3 is

$$\frac{1}{2}(2x_1 + x_2 + \delta_1)^2 + \frac{3}{2}(x_2 + \delta_2)^2 + c_{33} x_3^2, \quad (19)$$

where, from (15)

$$\delta_1 = a_{13} x_3, \quad \frac{1}{2}\delta_1 + \frac{3}{2}\delta_2 = a_{23} x_3.$$

To determine the coefficients a_{13} , a_{23} of an extreme form we give x_3 the minimum value 1 and use (17). In effect this gives

$$\delta_1 = p_1 x_3, \quad \delta_2 = (p_2 \pm \frac{1}{3})x_3$$

in (19), and the extreme forms are

$$\frac{1}{2}(2x_1 + x_2 + p_1 x_3)^2 + \frac{3}{2}\{x_2 + (p_2 \pm \frac{1}{3})x_3\}^2 + \frac{4}{3}x_3^2 \quad (20)$$

where $p_1 + p_2$ is odd.

All such forms are equivalent, for, if $\epsilon = \pm 1$, the unimodular substitution

$$\left. \begin{aligned} x_1 &\rightarrow x_1 + \frac{1}{2}(p'_1 - p'_2 - \epsilon p_1 + \epsilon p_2)x_3 \\ x_2 &\rightarrow x_2 + (p'_2 - \epsilon p_2)x_3 \\ x_3 &\rightarrow \epsilon x_3 \end{aligned} \right\}$$

transforms

$$2x_1 + x_2 + p_1 x_3, \quad x_2 + (p_2 \pm \frac{1}{3})x_3, \quad x_3$$

into

$$2x_1 + x_2 + p'_1 x_3, \quad x_2 + (p'_2 \pm \frac{1}{3}\epsilon)x_3, \quad \epsilon x_3;$$

and the substitution has integer coefficients when $p_1 + p_2$ and $p'_1 + p'_2$

are both odd (or both even). For a simple extreme form I take $p_1 = 1$, $p_2 = 0$ and the plus sign before $\frac{1}{3}$, getting

$$\begin{aligned}\Phi_3 &= 2(x_1^2 + x_2^2 + x_3^2) + 2(x_2x_3 + x_3x_1 + x_1x_2) \\ &= (x_2 + x_3)^2 + (x_3 + x_1)^2 + (x_1 + x_2)^2.\end{aligned}\quad (21)$$

4. Variation of x_1, x_2, x_3

When $n > 3$ and x_4, \dots, x_n are not all zero, I proceed as in § 3; after this I shall take the procedure for granted without further explanation. Fix the minimum of $G_3(x_3, \dots, x_n)$ at $\frac{1}{3}$ and, in virtue of Lemma 1, suppose this minimum attained at the unit point $x_3 = 1$, $x_4, \dots, x_n = 0$. Then $c_{33}, \delta_1, \delta_2$ retain their values from § 3.3, and in the extreme forms of F the terms independent of x_4, \dots, x_n give one of the extreme forms obtained in § 3.4. Sufficiently we take

$$\Phi_3 = \xi_1^2 + \xi_2^2 + \xi_3^2,$$

where, as in (21),

$$\xi_1 = x_2 + x_3, \quad \xi_2 = x_3 + x_1, \quad \xi_3 = x_1 + x_2. \quad (22)$$

Collecting all the terms involving x_1, x_2, x_3 , I write, if possible,

$$F = (\xi_1 + \delta_1)^2 + (\xi_2 + \delta_2)^2 + (\xi_3 + \delta_3)^2 + G_4(x_4, \dots, x_n), \quad (23)$$

where, identically in x_1, x_2, x_3 ,

$$\delta_1 \xi_1 + \delta_2 \xi_2 + \delta_3 \xi_3 = u_1 x_1 + u_2 x_2 + u_3 x_3$$

when

$$u_r = \sum_{s=1}^3 a_{rs} x_s \quad (r = 1, 2, 3).$$

This gives

$$u_1 = \delta_2 + \delta_3, \quad u_2 = \delta_3 + \delta_1, \quad u_3 = \delta_1 + \delta_2,$$

the system contragredient to (22), and the u are linearly independent functions of the δ since the ξ are linearly independent in x . We can therefore find δ to give the form (23) without restriction on the u . The argument is perfectly general: the ξ_1, \dots, ξ_r that we shall find will be evidently independent since their determinant will be a factor of Δ_r , and the corresponding u_1, \dots, u_r , being cogredient in the $\delta_1, \dots, \delta_r$, will therefore also be linearly independent. The form analogous to (23) will accordingly be always legitimate.

From (23) we have

$$\Delta(F) = \Delta_3 \Delta(G_4) = 4\Delta(G_4).$$

4.1. Separating the δ into integer and fractional parts as

$$\delta_r = p_r \pm \theta_r \quad (0 \leq \theta_r \leq \frac{1}{2}; \quad r = 1, 2, 3),$$

I consider the minimum of

$$\Theta_3 = \sum_{r=1}^3 (\xi_r + p_r \pm \theta_r)^2$$

for variation of x_1, x_2, x_3 . If the minima $\theta_1^2, \theta_2^2, \theta_3^2$ are attained simultaneously, Θ_3 has what I shall call its 'over-all' minimum $\sum_3 \theta^2$.

In this case the simultaneous equations

$$\xi_r + p_r = 0 \quad (r = 1, 2, 3)$$

have integer solutions in x , the necessary and sufficient condition for which is that $\sum p$ be even. Accordingly, when F is such that $\sum p$ is odd, the minimum of Θ_3 is greater (or at least as great) and $\Delta(G_4)$ is smaller (or at least as small). With this condition on $\sum p$ the equations

$$\xi_r + p_r = 0, 0, \pm 1 \quad (\text{in any order})$$

have integer solutions, and $\min \Theta_3$ is then the least of

$$(1 - \theta_\alpha)^2 + \theta_\beta^2 + \theta_\gamma^2 = 1 - 2\theta_\alpha + \sum_3 \theta^2,$$

where α, β, γ are 1, 2, 3 in some order. The least of these expressions is given by the greatest θ , so that, defining

$$\theta_\alpha = \max(\theta_1, \theta_2, \theta_3),$$

we have

$$\min \Theta_3 = 1 - 2\theta_\alpha + \sum_3 \theta^2.$$

To determine the greatest $\min \Theta_3$ (for variation of the θ) we have, since $\theta \leq \theta_\alpha$,

$$\min \Theta_3 \leq 1 - 2\theta_\alpha + 3\theta_\alpha^2 = 1 - 3\theta_\alpha(\frac{2}{3} - \theta_\alpha) \leq 1,$$

since $0 \leq \theta_\alpha \leq \frac{1}{2}$. Thus

$$\min \Theta_3 \leq 1$$

with equality only when all $\theta = \theta_\alpha$ and $\theta_\alpha = 0$, i.e. when $\theta_1, \theta_2, \theta_3 = 0$.

To sum up, $\min \Theta_3$ has its greatest value 1 when $\sum p$ is odd and every θ is 0. Then

$$G_4 = F - \Theta_3 \geq 2 - 1 = 1.$$

4.2. When $n = 4$ and $x_4 \neq 0$, we have, as before,

$$G_4 = c_{44} x_4^2 \geq 1,$$

$$\Delta(F_4) = \Delta_3 \Delta(G_4) = 4c_{44}.$$

Thus

$$\Delta_4 = \min \Delta(F_4) = 4. \quad (24)$$

The minimum of G_4 is given by $x_4 = 1$ and, arguing as in § 3.4, we obtain the extreme forms as

$$(\xi_1 + p_1 x_4)^2 + (\xi_2 + p_2 x_4)^2 + (\xi_3 + p_3 x_4)^2 + x_4^2. \quad (25)$$

These extreme forms are equivalent, for the unimodular substitution

$$\left. \begin{aligned} x_r &\rightarrow x_r + \{(p_r - p'_r) + \tfrac{1}{2} \sum (p' - p)\} x_4 \quad (r = 1, 2, 3) \\ x_4 &\rightarrow x_4 \end{aligned} \right\},$$

which has integer coefficients when $\sum p$ and $\sum p'$ are both odd (or both even), gives

$$\xi_r + p_r x_4 \rightarrow \xi'_r + p'_r x_4,$$

and so interchanges two different forms (23).*

I take $p_1 = 1, p_2, p_3 = 0$, giving as a representative extreme form

$$\Phi_4 = (x_2 + x_3 + x_4)^2 + (x_3 + x_1)^2 + (x_1 + x_2)^2 + x_4^2. \quad (26)$$

5. Variation of x_1, x_2, x_3, x_4

When $n > 4$ and x_5, \dots, x_n are not all zero, we proceed as in § 4, letting $G_4(x_4, \dots, x_n)$ attain the minimum 1 at the unit point $x_4 = 1, x_5, \dots, x_n = 0$. Suppressing the intermediate steps, which are still those of § 4, I consider

$$F = \sum_{r=1}^4 (\xi_r + p_r \pm \theta_r)^2 + G_5(x_5, \dots, x_n),$$

where, from (26) with a slight rearrangement,

$$\xi_1 = x_2 + x_3 + x_4, \quad \xi_2 = x_1 + x_2, \quad \xi_3 = x_1 + x_3, \quad \xi_4 = x_4.$$

Here again $\sum \xi$ is even, and

$$\Theta_4 = \sum_{r=1}^4 (\xi_r + p_r \pm \theta_r)^2$$

attains its 'over-all' minimum $\sum_4 \theta^2$ only if $\sum p$ is even. When $\sum p$ is odd, $\min \Theta_4$ is greater (or at least as great) and is one of

$$(1 - \theta_\alpha)^2 + \theta_\beta^2 + \theta_\gamma^2 + \theta_\delta^2 \quad (\alpha, \beta, \gamma, \delta = 1, 2, 3, 4).$$

More precisely, if we write

$$\theta_\alpha = \max(\theta_1, \theta_2, \theta_3, \theta_4),$$

then

$$\min \Theta_4 = 1 - 2\theta_\alpha + \sum_4 \theta^2.$$

* It should perhaps be pointed out that this proof of the equivalence of alternative extreme forms is essential to the argument. If there were non-equivalent extreme forms (in m variables say), they would have to be examined separately, as the induction proceeded to $m+1$ variables. For, though they gave the same Δ_m , they could not *a priori* be relied on to give the same Δ_{m+1} .

To determine the greatest $\min \Theta_4$ when the θ vary we have

$$\min \Theta_4 \leq 1 - 2\theta_\alpha + 4\theta_\alpha^2 = 1 - 4\theta_\alpha(\frac{1}{2} - \theta_\alpha) \leq 1,$$

since $0 \leq \theta_\alpha \leq \frac{1}{2}$. Thus

$$\min \Theta_4 \leq 1$$

with equality only when all $\theta = \theta_\alpha$ and $\theta_\alpha = 0, \frac{1}{2}$.

Thus $\min \Theta_4$ has its greatest value 1 (i) when $\sum p$ is odd and all $\theta = 0$; or (ii) when all $\theta = \frac{1}{2}$. In (ii) the condition on $\sum p$ is dropped since the same minimum is equally given by the over-all minimum, $\sum \theta^2$. As usual we have

$$G_5 = F - \Theta_4 \geq 2 - 1 = 1.$$

5.1. When $n = 5$ and $x_5 \neq 0$, we have, proceeding as before,

$$\begin{aligned} \min \Delta(G_5) &= 1, & \Delta(F_5) &= \Delta_4 \Delta(G_5) = 4\Delta(G_5), \\ \Delta_5 &= \min \Delta(F_5) = 4. \end{aligned} \quad (27)$$

The extreme forms are of two types

$$(i) \sum_{r=1}^4 (\xi_r + p_r x_5)^2 + x_5^2, \quad (ii) \sum_{r=1}^4 \{\xi_r + (p_r + \frac{1}{2})x_5\}^2 + x_5^2.$$

In (i) $\sum p$ is odd; in (ii) $\sum p$ may be odd or even. By a substitution similar to that used in §4.2 we can see that two forms (i) are equivalent, and that two forms (ii) are equivalent if $\sum p$ has the same parity in both.

The unimodular integer substitution

$$(x_1, x_2, x_3, x_4, x_5) \rightarrow (x_1 - x_4, x_2 + x_4, x_3 + x_4, -x_4, x_5)$$

leaves ξ_1, ξ_2, ξ_3, x_5 unaltered and changes x_4 , i.e. ξ_4 , only in sign. In (ii) this has the effect of changing the sign of $p_4 + \frac{1}{2}$, i.e. of replacing p_4 by $-(p_4 + 1)$, which changes the parity of $\sum p$. Thus all forms (ii) are equivalent.

To interconnect the types (i) and (ii) it is now enough to relate one form of each type. In (i) take $p_1 = 1, p_2, p_3, p_4 = 0$ to give

$$\Phi_5 = (x_2 + x_3 + x_4 + x_5)^2 + (x_1 + x_2)^2 + (x_1 + x_3)^2 + x_4^2 + x_5^2. \quad (28)$$

In (ii) take $p_1, p_2, p_3 = 0, p_4 = -1$ and transform the \sum by the identity

$$\sum_1^4 a_r^2 = \sum_{r=1}^4 (s - a_r)^2 \quad \left(s = \frac{1}{2} \sum_{r=1}^4 a_r\right).$$

Noting that $s = x_1 + x_2 + x_3 + x_4 + \frac{1}{2}x_5$, we find that the transformation gives (28) with x_1, x_4 interchanged. All extreme forms are thus seen to be equivalent. I select (28) as my working form.

6. Variation of x_1, \dots, x_5

When $n > 5$ and x_6, \dots, x_n are not all zero, the argument proceeds as before, and we consider

$$F = \sum_{r=1}^5 (\xi_r + p_r \pm \theta_r)^2 + G_6(x_6, \dots, x_n),$$

where the ξ are those that occur in (28), namely

$$\left. \begin{aligned} \xi_1 &= x_2 + x_3 + x_4 + x_5, & \xi_2 &= x_1 + x_2, & \xi_3 &= x_1 + x_3, \\ \xi_4 &= x_4, & \xi_5 &= x_5. \end{aligned} \right\} \quad (29)$$

Here again $\sum \xi$ is even, and

$$\Theta_5 = \sum_{r=1}^5 (\xi_r + p_r \pm \theta_r)^2$$

attains its over-all minimum $\sum \theta^2$ only if $\sum p$ is even, since then and only then the equations

$$\xi_r + p_r = 0 \quad (r = 1, \dots, 5)$$

have integer solutions in x . As in §§ 4, 5, when $\sum p$ is odd, $\min \Theta_5$ is greater (or as great), having the value

$$1 - 2\theta_\alpha + \sum \theta^2,$$

where $\theta_\alpha = \max(\theta_1, \dots, \theta_5)$. Thus, arguing as before,

$$\min \Theta \leq 1 - 2\theta_\alpha + 5\theta_\alpha^2 = \frac{5}{4} - (\frac{1}{2} - \theta_\alpha)(\frac{1}{2} + 5\theta_\alpha) \leq \frac{5}{4},$$

since $0 \leq \theta_\alpha \leq \frac{1}{2}$. Thus

$$\min \Theta_5 \leq \frac{5}{4},$$

with equality only when all $\theta = \theta_\alpha$ and $\theta_\alpha = \frac{1}{2}$. This minimum is equally the over-all minimum $\sum \theta^2$ when $\theta = \frac{1}{2}$. We may therefore discard the condition on $\sum p$ relying solely on the condition that all $\theta = \frac{1}{2}$. Then

$$G_6 = F - \Theta_5 \geq 2 - \frac{5}{4} = \frac{3}{4}.$$

6.1. When $n = 6$ and $x_6 \neq 0$, we therefore have

$$\begin{aligned} \min \Delta(G_6) &= \frac{3}{4}, & \Delta(F_6) &= \Delta_5 \Delta(G_6) = 4\Delta(G_6), \\ \Delta_6 &= \min \Delta(F_6) = 3. \end{aligned} \quad (30)$$

The extreme forms are

$$\sum_{r=1}^5 \{\xi_r + (p_r + \frac{1}{2})x_6\}^2 + \frac{3}{4}x_6^2,$$

where $\sum p$ may be even or odd. A transformation similar to that of

§ 4.2 shows that two such forms are equivalent if $\sum p$ has the same parity in each. If not, we change the sign of x_6 in one of the forms. This effectively replaces p_r by $-(p_r+1)$, and so changes the parity of $\sum p_r$ since \sum contains an odd number of terms. Thus all extreme forms are equivalent. I take every p zero to give the typical form

$$\Phi_6 = (x_2+x_3+x_4+x_5+\frac{1}{2}x_6)^2 + (x_1+x_2+\frac{1}{2}x_6)^2 + (x_1+x_3+\frac{1}{2}x_6)^2 + \\ + (x_4+\frac{1}{2}x_6)^2 + (x_5+\frac{1}{2}x_6)^2 + \frac{3}{4}x_6^2. \quad (31)$$

7. Variation of x_1, \dots, x_6

When $n > 6$ and x_7, \dots, x_n are not all zero, we may therefore sufficiently consider

$$F = \Theta_6 + G_7(x_7, \dots, x_n),$$

where
$$\Theta_6 = \sum_1^5 (\xi_r + \frac{1}{2}x_6 + p_r \pm \theta_r)^2 + 3(\frac{1}{2}x_6 + p_6 \pm \theta_6)^2$$

and the ξ are those given in (29). Here Θ_6 will attain its over-all minimum $\sum \theta^2 + 3\theta_6^2$ if the equations

$$\eta_r = \xi_r + \frac{1}{2}x_6 + p_r = 0 \quad (r = 1, \dots, 5),$$

$$\eta_6 = \frac{1}{2}x_6 + p_6 = 0$$

have integer solutions in x . Then x_6 is even and $\sum p \left(= -\sum_1^5 \xi - 3x_6 \right)$ is even since $\sum \xi$ is even. Moreover, this condition on $\sum p$ is sufficient; and, more generally, the equations $\eta_r = q_r$ ($r = 1, \dots, 6$), where the q are any integers, have integer solutions in x when $\sum p, \sum q$ have the same parity. Thus, as usual, $\min \Theta_6$ is greater or as great when $\sum p$ is odd. With this condition $\min \Theta_6$ for even values of x_6 is attained when every η except one (say η_s) is zero and η_s itself is unity with a sign opposite to that of θ_s . Separating the case in which η_s is η_6 , we see that $\min \Theta_6$, when x_6 is even, is either

$$\omega_1 = \sum_1^5 \theta^2 + 3(1 - \theta_6)^2$$

or

$$\omega_2 = 1 - 2\theta_\alpha + \sum_1^5 \theta^2 + 3\theta_6^2,$$

where, as in § 6,

$$\theta_\alpha = \max(\theta_1, \dots, \theta_5).$$

7.1. When x_6 is odd, the over-all minimum, i.e. that in which the minima of the several terms occur simultaneously, is

$$\sum_1^5 (\frac{1}{2} - \theta_r)^2 + 3(\frac{1}{2} - \theta_6)^2. \quad (32)$$

Write $\epsilon_r = \pm \frac{1}{2}$, the sign being opposite that of the corresponding θ_r .

Then the minimum (32) is attained when $\eta_r = \epsilon_r$ ($r = 1, \dots, 6$). This gives

$$\sum \epsilon = \sum \eta = \sum \xi + 3x_6 + \sum p,$$

and therefore $\sum \epsilon$ is even since $\sum \xi$ is even and $3x_6, \sum p$ are odd. This requires an odd number of minus (and of plus) signs in the ϵ and therefore also in the θ . Accordingly we exclude the minimum (32) and give $\min \Theta_6$ a greater (or equal) value by taking an even number (0, 2, 4, or 6) of θ with plus signs. In this case Θ_6 attains its minimum when every η but one (say η_s) equals the corresponding ϵ , but η_s is $-\epsilon_s$. Separating, as above, the case in which η_s is η_6 , we see that $\min \Theta_6$ when x_6 is odd is either

$$\omega_3 = \sum_5 (\tfrac{1}{2} - \theta)^2 + 3(\tfrac{1}{2} + \theta_6)^2$$

or one of

$$\begin{aligned} \omega_4 &= (\tfrac{1}{2} + \theta_\beta)^2 + \sum_{r \neq \beta} (\tfrac{1}{2} - \theta_r)^2 + 3(\tfrac{1}{2} - \theta_6)^2 \\ &= 2\theta_\beta + \sum_5 (\tfrac{1}{2} - \theta)^2 + 3(\tfrac{1}{2} - \theta_6)^2, \end{aligned}$$

where $\min \omega_4$ occurs when

$$\theta_\beta = \min(\theta_1, \dots, \theta_5).$$

With this interpretation of $\theta_\alpha, \theta_\beta$ as the greatest and least of $\theta_1, \dots, \theta_5$ we have

$$\min \Theta_6 = \min(\omega_1, \omega_2, \omega_3, \omega_4).$$

7.2. We now consider the greatest value of this $\min \Theta_6$ for variation of θ . After reduction we have

$$\begin{aligned} \omega_2 + \omega_4 &= 3 - 2\theta_\alpha + 2\theta_\beta - \sum_5 \theta + 2 \sum_5 \theta^2 - 3\theta_6 + 6\theta_6^2 \\ &= \tfrac{8}{3} - 2(\theta_\alpha - \theta_\beta) - 2 \sum_5 \theta(\tfrac{1}{2} - \theta) + 6(\theta_6 - \tfrac{1}{3})(\theta_6 - \tfrac{1}{6}). \end{aligned} \quad (33)$$

But $\theta_\alpha \geq \theta_\beta$ and $0 \leq \theta \leq \tfrac{1}{2}$, and so, unless either $\theta_6 > \tfrac{1}{3}$ or $< \tfrac{1}{6}$,

$$\omega_2 + \omega_4 \leq \tfrac{8}{3},$$

and therefore

$$\min(\omega_2, \omega_4) \leq \tfrac{4}{3}.$$

I define here the 'reversing transformation'

$$\theta_r \rightarrow \tfrac{1}{2} - \theta_r \quad (r = 1, \dots, 6). \quad (34)$$

The effect of this transformation is to interchange ω_1 with ω_3 and ω_2 with ω_4 , and so to leave $\min(\omega_1, \omega_2, \omega_3, \omega_4)$ unaltered. It leaves unaltered the inequalities $0 \leq \theta_r \leq \tfrac{1}{2}$ and interchanges the two

conditions $\theta_6 > \frac{1}{3}$ and $\theta_6 < \frac{1}{3}$. It is sufficient therefore to consider only $\theta_6 > \frac{1}{3}$, which gives $\frac{1}{2} - \theta_6 < \frac{1}{6}$. Thus

$$\omega_4 < \frac{4}{3} + 2\theta_6 - \sum_5 \theta + \sum_5 \theta^2.$$

But $\theta_\beta \leq \frac{1}{5} \sum \theta$, since θ_β is the least θ . Thus

$$\omega_4 < \frac{4}{3} - \sum_5 \theta(\frac{2}{3} - \theta) \leq \frac{4}{3},$$

since $\theta(\frac{2}{3} - \theta) \geq 0$. Hence

$$\omega_4 < \frac{4}{3}, \text{ if } \theta_6 > \frac{1}{3}.$$

If $\theta_6 = \frac{1}{3}$ and $\theta_r = 0$ ($r = 1, \dots, 5$), then $\omega_4 = \frac{4}{3}$, and, from (33), $\omega_2 + \omega_4 = \frac{8}{3}$. Thus

$$\min(\omega_2, \omega_4) \leq \frac{4}{3},$$

with equality only if

$$(i) \quad \theta_r = 0 \quad (r = 1, \dots, 5), \quad \theta_6 = \frac{1}{3}$$

or (on reversal)

$$(ii) \quad \theta_r = \frac{1}{2} \quad (r = 1, \dots, 5), \quad \theta_6 = \frac{1}{6}.$$

With (i) we have also $\omega_1 = \frac{4}{3}$, $\omega_3 = \frac{10}{3}$, and so $\min(\omega_1, \omega_2, \omega_3, \omega_4) = \frac{4}{3}$.

Thus $\frac{4}{3}$ is the greatest minimum of Θ_6 and is given either by (i) or by (ii) and in each case by $\sum p$ odd. Then

$$G_7 = F - \Theta_6 \geq 2 - \frac{4}{3} = \frac{2}{3}.$$

7.3. When $n = 7$ and $x_7 \neq 0$, we therefore have

$$\begin{aligned} \min \Delta(G_7) &= \frac{2}{3}, & \Delta(F_7) &= \Delta_6 \Delta(G_7) = 3\Delta(G_7), \\ \Delta_7 &= \min \Delta(F_7) = 2. \end{aligned} \tag{35}$$

In considering the extreme forms we recall the further condition that in the signs of the θ there must be even numbers of plus and minus. In (i), the signs of $\theta_r = 0$ being indeterminate, we may have $\pm \theta_6$ which gives

$$(i) \quad \sum_{r=1}^5 (\xi_r + \frac{1}{2}x_6 + p_r x_7)^2 + 3\{\frac{1}{2}x_6 + (p_6 \pm \frac{1}{3})x_7\}^2 + \frac{2}{3}x_7^2.$$

In (ii) minus signs before θ_r give rise to $p_r - \frac{1}{2}$ as coefficients of x_7 . These, as always, I prefer to rewrite as $(p_r - 1) + \frac{1}{2}$, thus changing at once the sign before θ_r and the parity of $\sum p_r$, since $p_r - 1$ replaces p_r . We thus obtain the second type of extreme form

$$(ii) \quad \sum_{r=1}^5 \{\xi_r + \frac{1}{2}x_6 + (p_r + \frac{1}{2})x_7\}^2 + 3\{\frac{1}{2}x_6 + (p_6 \pm \frac{1}{3})x_7\}^2 + \frac{2}{3}x_7^2,$$

where now the upper or lower sign is taken before $\frac{1}{2}$ according as $\sum p$ is odd or even. In (i), on the other hand, the sign after p_6 is arbitrary and $\sum p$ is always odd.

7.4. The usual form of substitution then shows that two forms of the same type and with the same sign following p_6 are equivalent, since $\sum p$ will have the same parity in the two forms. To connect two forms of the same type with opposite signs after p_6 we merely change the sign of x_7 . This reverses the double sign after p_6 and changes the sign of every p . This leaves the parity of $\sum p$ unaltered in (i) and changed in (ii) which is right.

If we replace x_6 by $x_6 + x_7$, we change (i) with the lower sign into (ii) with the upper sign. All the extreme forms are consequently equivalent; I take $p_1, \dots, p_5 = 0, p_6 = 1$ and the lower sign in (i) to give

$$\sum_{r=1}^5 (\xi_r + \frac{1}{2}x_6)^2 + 3(\frac{1}{2}x_6 + \frac{2}{3}x_7)^2 + \frac{2}{3}x_7^2.$$

The final pair of terms can be rearranged into

$$\frac{1}{4}x_6^2 + 2(x_7 + \frac{1}{2}x_6)^2,$$

and so, written out, we have the form

$$\begin{aligned} \Phi_7 = & (x_2 + x_3 + x_4 + x_5 + \frac{1}{2}x_6)^2 + (x_1 + x_2 + \frac{1}{2}x_6)^2 + (x_1 + x_3 + \frac{1}{2}x_6)^2 + \\ & + (x_4 + \frac{1}{2}x_6)^2 + (x_5 + \frac{1}{2}x_6)^2 + (\frac{1}{2}x_6)^2 + 2(x_7 + \frac{1}{2}x_6)^2, \end{aligned} \quad (36)$$

which I shall regard as the typical extreme form in seven variables.

8. Variation of x_1, \dots, x_7

When $n > 7$ and x_8, \dots, x_n are not all zero, we sufficiently consider

$$F = \Theta_7 + G_8(x_8, \dots, x_n),$$

where

$$\Theta_7 = \sum_{r=1}^5 (\xi_r + \frac{1}{2}x_6 + p_r \pm \theta_r)^2 + (\frac{1}{2}x_6 + p_6 \pm \theta_6)^2 + 2(x_7 + \frac{1}{2}x_6 + p_7 \pm \theta_7)^2$$

and the ξ are again those of (29). We argue as in the preceding section. Θ_7 attains its over-all minimum $\sum \theta^2 + 2\theta_7^2$ if the equations

$$\begin{aligned} \eta_r = \xi_r + \frac{1}{2}x_6 + p_r &= 0 \quad (r = 1, \dots, 5), & \eta_6 = \frac{1}{2}x_6 + p_6 &= 0, \\ \eta_7 = x_7 + \frac{1}{2}x_6 + p_7 &= 0 \end{aligned}$$

have integer solutions in x . Then, as before, x_6 is even and $\sum_6 p$ ($= -\sum_5 \xi - 3x_6$) is even. Since x_7 occurs only in the last equation, this equation is always soluble in integers, when x_6 is even, without any condition on p_7 .

Thus, when $\sum_{\mathbb{I}} p$ is odd, the over-all minimum of Θ_7 is excluded, and $\min \Theta_7$, when x_6 is even, is

$$\rho_1 = 1 - 2\theta_\alpha + \sum_{\mathbb{I}} \theta^2 + 2\theta_7^2,$$

where $\theta_\alpha = \max(\theta_1, \dots, \theta_6)$. This is analogous to the ω_1, ω_2 of § 7 without the need of distinguishing between them.

8.1. When x_6 is odd, the corresponding over-all minimum, analogous to (32) above, is

$$\sum_{\mathbb{I}} (\tfrac{1}{2} - \theta)^2 + 2(\tfrac{1}{2} - \theta_7)^2. \quad (37)$$

Again define $\epsilon_r = \pm \tfrac{1}{2}$ where the sign is opposite to that of the corresponding θ_r . The minimum (37) is attained when $\eta_r = \epsilon_r$ ($r = 1, \dots, 7$). The equation $\eta_7 = \epsilon_7$ is always soluble in integers, when x_6 is odd, since x_7 is always at our disposal. For the rest

$$\sum_{\mathbb{I}} \epsilon = \sum_{\mathbb{I}} \eta = \sum \xi + 3x_6 + \sum_{\mathbb{I}} p,$$

as before, and so, again, $\sum_{\mathbb{I}} \epsilon$ is even. This requires an odd number of minus and of plus signs in the ϵ and so in the θ . We thus exclude the over-all minimum (37) by taking an even number of θ with plus signs: this gives Θ_7 the minimum (when x_6 is odd)

$$\rho_2 = 2\theta_\beta + \sum_{\mathbb{I}} (\tfrac{1}{2} - \theta)^2 + 2(\tfrac{1}{2} - \theta_7)^2$$

where $\theta_\beta = \min(\theta_1, \dots, \theta_6)$. This corresponds to ω_3, ω_4 above without the need to distinguish between them. Thus, for general x_6 ,

$$\min \Theta_7 = \min(\rho_1, \rho_2).$$

8.2. It remains to consider the greatest value of this $\min \Theta_7$ as the θ vary. We have, after reduction,

$$\rho_1 + \rho_2 = 3 - 2(\theta_\alpha - \theta_\beta) - 2 \sum_{\mathbb{I}} \theta(\tfrac{1}{2} - \theta) - 4\theta_7(\tfrac{1}{2} - \theta_7) \leq 3,$$

since $\theta_\alpha \geq \theta_\beta$ and $0 \leq \theta \leq \tfrac{1}{2}$. Thus

$$\min(\rho_1, \rho_2) \leq \tfrac{3}{2},$$

and equality is not possible unless $\theta_\alpha = \theta_\beta$, i.e. $\theta_1, \dots, \theta_6$ are all equal, and every θ is 0 or $\tfrac{1}{2}$.

The reversing transformation

$$\theta \rightarrow \tfrac{1}{2} - \theta$$

interchanges ρ_1 and ρ_2 and so leaves $\min(\rho_1, \rho_2)$ unaltered. It is therefore sufficient to take $\theta_7 = \frac{1}{2}$. This gives

$$\begin{aligned}\rho_2 &= \frac{3}{2} + 2\theta_\beta - \sum_6 \theta + \sum_6 \theta^2 \\ &\leq \frac{3}{2} - \sum_6 \theta(\frac{3}{2} - \theta),\end{aligned}$$

since $\theta_\beta \leq \frac{1}{6} \sum \theta$. Thus $\rho_2 \leq \frac{3}{2}$, with equality only when all $\theta = 0$. Hence $\min(\rho_1, \rho_2)$ has its greatest value $\frac{3}{2}$

$$(i) \text{ when } \theta_r = 0 \ (r = 1, \dots, 6), \quad \theta_7 = \frac{1}{2};$$

or, by the reversing substitution,

$$(ii) \text{ when } \theta_r = \frac{1}{2} \ (r = 1, \dots, 6), \quad \theta_7 = 0.$$

$$\text{Thus } G_8 = F - \Theta_7 \geq 2 - \frac{3}{2} = \frac{1}{2}.$$

8.3. When $n = 8$ and $x_8 \neq 0$ we therefore have

$$\begin{aligned}\min \Delta(G_8) &= \frac{1}{2}, \quad \Delta(F_8) = \Delta_7 \Delta(G_8) = 2\Delta(G_8), \\ \Delta_8 &= \min \Delta(F_8) = 1.\end{aligned}\tag{38}$$

The extreme forms corresponding to the values of θ given by (i) and (ii) are of the two types

$$\begin{aligned}(i) \quad &\sum_{r=1}^5 (\xi_r + \frac{1}{2}x_6 + p_r x_8)^2 + (\frac{1}{2}x_6 + p_6 x_8)^2 + \\ &+ 2\{x_7 + \frac{1}{2}x_6 + (p_7 + \frac{1}{2})x_8\}^2 + \frac{1}{2}x_8^2, \\ (ii) \quad &\sum_{r=1}^5 \{\xi_r + \frac{1}{2}x_6 + (p_r + \frac{1}{2})x_8\}^2 + \{\frac{1}{2}x_6 + (p_6 + \frac{1}{2})x_8\}^2 + \\ &+ 2(x_7 + \frac{1}{2}x_6 + p_7 x_8)^2 + \frac{1}{2}x_8^2.\end{aligned}$$

In (i) the condition on the signs of the θ disappears since the relevant θ are zero. In (ii) $-\theta_r$ gives $p_r - \frac{1}{2}$, which we write $(p_r - 1) + \frac{1}{2}$, changing the parity of $\sum_6 p$. Since the number of such minus signs is even, this finally leaves the parity of $\sum_6 p$ unchanged. Hence both (i), (ii) are subject to the condition that $\sum_6 p$ is odd.

Two forms of the same type are equivalent if p_7 has the same parity in both, for then $\sum_7 p$ has the same parity in both, and the usual arguments apply. If in (i) we change the sign of x_8 , we effectively change the parity of p_7 leaving that of $\sum_6 p$ unaltered. Thus all forms (i) are equivalent.

If in (i) we replace x_6 by $x_6 + x_8$, we obtain (ii) with $p_7 + 1$ in place of p_7 . Thus all the extreme forms are equivalent. For a typical form I take $p_1, \dots, p_5, p_7 = 0, p_6 = 1$ in (i), getting

$$\sum_{r=1}^5 (\xi_r + \frac{1}{2}x_6)^2 + (\frac{1}{2}x_6 + x_8)^2 + 2(x_7 + \frac{1}{2}x_6 + \frac{1}{2}x_8)^2 + \frac{1}{2}x_8^2.$$

The sum of the last three terms can be rewritten

$$(x_7 + \frac{1}{2}x_6)^2 + (x_8 + \frac{1}{2}x_6)^2 + (x_7 + x_8 + \frac{1}{2}x_6)^2,$$

so that at length

$$\begin{aligned} \Phi_8 = & (x_2 + x_3 + x_4 + x_5 + \frac{1}{2}x_6)^2 + (x_1 + x_2 + \frac{1}{2}x_6)^2 + \\ & + (x_1 + x_3 + \frac{1}{2}x_6)^2 + (x_4 + \frac{1}{2}x_6)^2 + (x_5 + \frac{1}{2}x_6)^2 + \\ & + (x_7 + \frac{1}{2}x_6)^2 + (x_8 + \frac{1}{2}x_6)^2 + (x_7 + x_8 + \frac{1}{2}x_6)^2. \end{aligned} \quad (39)$$

9. Variation of x_1, \dots, x_8

When $n > 8$ and x_9, \dots, x_n are not all zero, we can therefore consider

$$F = \Theta_8 + G_9(x_9, \dots, x_n),$$

where

$$\Theta_8 = \sum_{r=1}^8 (\xi_r + \frac{1}{2}x_6 + p_r \pm \theta_r)^2$$

and the ξ are derived from (39). Explicitly

$$\left. \begin{aligned} \xi_1 &= x_2 + x_3 + x_4 + x_5, & \xi_2 &= x_1 + x_2, & \xi_3 &= x_1 + x_3 \\ \xi_4 &= x_4, & \xi_5 &= x_5, & \xi_6 &= x_7, & \xi_7 &= x_8 \\ \xi_8 &= x_7 + x_8. \end{aligned} \right\} \quad (40)$$

As in the preceding sections, Θ_8 attains its over-all minimum $\sum_8 \theta^2$ if the equations

$$\eta_r = \xi_r + \frac{1}{2}x_6 + p_r = 0 \quad (r = 1, \dots, 8)$$

have integer solutions in x . The necessary and sufficient condition is that $\sum_8 p$ be even, and the solution, incidentally, gives x_6 even.

To see this write $\zeta_r = \xi_r + \frac{1}{2}x_6$; then it can be verified that

$$\left. \begin{aligned} x_1 &= -\zeta_1 - 2\zeta_6 - 2\zeta_7 + \zeta_8 + \frac{1}{2} \sum \zeta, & x_2 &= \zeta_1 + \zeta_2 + \zeta_6 + \zeta_7 - \frac{1}{2} \sum \zeta, \\ x_3 &= \zeta_1 + \zeta_2 + \zeta_6 + \zeta_7 - \frac{1}{2} \sum \zeta, & x_4 &= \zeta_4 - \zeta_6 - \zeta_7 + \zeta_8, \\ x_5 &= \zeta_5 - \zeta_6 - \zeta_7 + \zeta_8, & x_6 &= 2(\zeta_6 + \zeta_7 - \zeta_8), \\ x_7 &= -\zeta_7 + \zeta_8, & x_8 &= -\zeta_6 + \zeta_8, \end{aligned} \right\} \quad (41)$$

where $\sum \zeta$ always means $\sum_8 \zeta$. Thus, when the ζ are integers, so too are the x so long as $\sum \zeta$ is even. The equations $\eta = 0$ give $\zeta = -p$

and so the corresponding x are integer when $\sum p$ is even. Thus, when $\sum p$ is odd, the over-all minimum is excluded, and $\min \Theta_8$, when x_6 is even, is

$$\rho_1 = 1 - 2\theta_\alpha + \sum_8 \theta^2, \quad (42)$$

where $\theta_\alpha = \max(\theta_1, \dots, \theta_8)$.

When x_6 is odd, the over-all minimum $\sum (\frac{1}{2} - \theta)^2$ is attained if the equations $\eta_r = \epsilon_r$ have integer solutions in x , where ϵ_r is defined as in the preceding sections. Then

$$\sum_8 \epsilon = \sum_8 \eta = \sum_8 \xi + 4x_6 + \sum_8 p,$$

so that $\sum \epsilon$ is odd since $\sum p$ is odd. Thus, when $\sum \epsilon$ is even, and x_6 is odd, Θ_8 has the greater (or equal) minimum

$$\rho_2 = 2\theta_\beta + \sum_8 (\frac{1}{2} - \theta)^2, \quad (43)$$

where $\theta_\beta = \min(\theta_1, \dots, \theta_8)$. This condition on ϵ requires that an even number of θ shall have plus signs. Thus, with the stated conditions on $\sum p$ and the signs of θ , we have

$$\min \Theta_8 = \min(\rho_1, \rho_2).$$

9.1. It remains to consider the greatest value of this minimum for variation of θ . I write $\theta = \frac{1}{4} + \phi$, so that now $|\phi| \leq \frac{1}{4}$ and ϕ may have either sign. I suppose the ϕ renumbered (if necessary) so that the numerical order of the suffixes is the order of descending ϕ , i.e. so that

$$\frac{1}{4} \geq \phi_1 \geq \phi_2 \geq \dots \geq \phi_7 \geq \phi_8 \geq -\frac{1}{4}.$$

Then

$$\rho_1 = 1 - 2\phi_1 + \frac{1}{2} \sum \phi + \sum \phi^2, \quad \rho_2 = 1 + 2\phi_8 - \frac{1}{2} \sum \phi + \sum \phi^2,$$

the summation \sum being over all ϕ . I consider in order the various possible distributions of sign among ϕ .

If all $\phi \geq 0$, we have $2\phi_8 \leq \frac{1}{2} \sum \phi$, and so

$$\rho_2 \leq 1 - \sum \phi (\frac{1}{4} - \phi) \leq 1$$

with equality only if (i) all $\phi = 0$ or (ii) all $\phi = \frac{1}{4}$. These give (i) $\rho_1 = 1$, (ii) $\rho_2 = 2$. Thus, when all $\phi \geq 0$,

$$\min(\rho_1, \rho_2) = 1.$$

If one ϕ is negative, it will be ϕ_8 . Write $\phi_8 = -\psi_8$, so that $\psi_8 > 0$. Then

$$\rho_2 = 1 - \sum_{r=1}^7 \phi_r (\frac{1}{2} - \phi_r) - \psi_8 (\frac{3}{2} - \psi_8) < 1.$$

With two negative ϕ write $\phi_7, \phi_8 = -\psi_7, -\psi_8$, so that $\psi_8 \geq \psi_7 > 0$. Then

$$\rho_2 = 1 - \sum_{r=1}^6 \phi_r \left(\frac{1}{2} - \phi_r\right) - \sum_{r=7}^8 \psi_r \left(\frac{1}{2} - \psi_r\right) - (\psi_8 - \psi_7) < 1.$$

When three ϕ are negative, write

$$\phi_6, \phi_7, \phi_8 = -\psi_6, -\psi_7, -\psi_8$$

with $\psi_8 \geq \psi_7 \geq \psi_6 > 0$, and consider

$$\begin{aligned} \frac{5}{12}\rho_1 + \frac{7}{12}\rho_2 &= 1 - \frac{5}{6}\phi_1 - \frac{7}{6}\psi_8 - \frac{1}{12} \sum_{r=1}^5 \phi_r + \frac{1}{12} \sum_{r=6}^8 \psi_r + \sum_{r=1}^5 \phi_r^2 + \sum_{r=6}^8 \psi_r^2 \\ &< 1 - \sum_{r=1}^5 \phi_r \left(\frac{1}{4} - \phi_r\right) - \sum_{r=6}^8 \psi_r \left(\frac{1}{4} - \psi_r\right) \end{aligned}$$

$$\text{since } \frac{5}{6}\phi_1 \geq \frac{1}{6} \sum_{r=1}^5 \phi_r, \quad \frac{7}{6}\psi_8 > \psi_8 \geq \frac{1}{3} \sum_{r=6}^8 \psi_r.$$

Since $0 \leq \phi, \psi \leq \frac{1}{4}$, we have

$$\frac{5}{12}\rho_1 + \frac{7}{12}\rho_2 < 1,$$

and so again $\min(\rho_1, \rho_2) < 1$.

Lastly, with four negative ϕ , writing $-\psi$ for these, we have

$$\begin{aligned} \frac{1}{2}(\rho_1 + \rho_2) &= 1 - \phi_1 - \psi_8 + \sum_{r=1}^4 \phi_r^2 + \sum_{r=5}^8 \psi_r^2 \\ &\leq 1 - \sum_{r=1}^4 \phi_r \left(\frac{1}{4} - \phi_r\right) - \sum_{r=5}^8 \psi_r \left(\frac{1}{4} - \psi_r\right) \leq 1, \end{aligned}$$

since $\phi_1 \geq \frac{1}{4} \sum_{r=1}^4 \phi_r$, $\psi_8 \geq \frac{1}{4} \sum_{r=5}^8 \psi_r$. There is equality only when all $\psi = \frac{1}{4}$ (we have specifically excluded $\psi = 0$) and (i) all $\phi = 0$ or (ii) all $\phi = \frac{1}{4}$. We then get (i) $\rho_1 = \frac{3}{4}$, (ii) $\rho_1 = 1$. Thus (i) gives a minimum less than 1; (ii) gives again $\min(\rho_1, \rho_2) = 1$.

If we change the sign of every ϕ (which is again the 'reversing transformation'), we interchange ρ_1, ρ_2 and therefore leave $\min(\rho_1, \rho_2)$ unaltered. We need not therefore consider the effect of more than four negative ϕ .

Thus the greatest value of $\min \Theta_8$ is 1, attained when (i) all $\theta = \frac{1}{4}$, (ii) all $\theta = \frac{1}{2}$, (iii) all $\theta = 0$ (by the reversing transformation), (iv) $\theta = (0, 0, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ in some order. Further $\sum p$ must be odd, and the numbers of θ with plus and with minus signs must both be even. Then

$$G_9 = F - \Theta_8 \geq 2 - 1 = 1.$$

9.2. When $n = 9$ and $x_9 \neq 0$, we consequently have

$$\min \Delta(G_9) = 1, \quad \Delta(F_9) = \Delta_8 \Delta(G_9) = \Delta(G_9),$$

$$\Delta_9 = \min \Delta(F_9) = 1, \quad (44)$$

and so

$$\gamma_9 = 2. \quad (45)$$

The extreme forms are of the four types

- (i) $\sum_{r=1}^8 \{\xi_r + \frac{1}{2}x_6 + (p_r \pm \frac{1}{4})x_9\}^2 + x_9^2;$
 (ii) $\sum_{r=1}^8 \{\xi_r + \frac{1}{2}x_6 + (p_r + \frac{1}{2})x_9\}^2 + x_9^2;$
 (iii) $\sum_{r=1}^8 (\xi_r + \frac{1}{2}x_6 + p_r x_9)^2 + x_9^2;$
 (iv) $\sum \{\xi_r + \frac{1}{2}x_6 + (p_r + \frac{1}{2})x_9\}^2 + \sum (\xi_s + \frac{1}{2}x_6 + p_s x_9)^2 + x_9^2,$

where, in (iv), r is taken over any four of the suffixes 1, ..., 8 and s over the other four. We have $\sum p$ odd throughout, and in (i) there are even numbers of plus and of minus signs; in (ii) the minus signs have been removed under an argument used in § 8.3.

9.3. In type (i) the general form is transformed into a general form with positive signs by a substitution

$$\left. \begin{aligned} \pm(\xi_r + \frac{1}{2}x_6 + p_r x_9) + \frac{1}{4}x_9 &= \xi'_r + \frac{1}{2}x'_6 + p'_r x'_9 + \frac{1}{4}x'_9 \quad (r = 1, \dots, 8) \\ x_9 &= x'_9. \end{aligned} \right\} \quad (46)$$

Thus, in the notation of (41),

$$\zeta'_r = \pm(\zeta_r + p_r x_9) - p'_r x_9. \quad (47)$$

Then

$$\frac{1}{2} \sum_8 \zeta' = \frac{1}{2} \sum_8 \zeta - \sum \zeta_s + \left(\frac{1}{2} \sum_8 (p - p') - \sum p_s \right) x_9,$$

where the suffix s refers to terms with minus sign in (47). Now in $\zeta (= \xi + \frac{1}{2}x_6)$ the coefficients of the x are not integers, but they are integers in the sum or difference of any two ζ . Thus the coefficients in the sum or difference of any two ζ' are integers; and so too in $\sum \zeta_s$, since there are an even number of minus signs in (i) and therefore in (47).

Again in $\frac{1}{2} \sum \zeta (= \sum x)$ the coefficients of the x are integers, and $\sum (p - p')$ is even since $\sum p$ and $\sum p'$ are both odd. Thus the coefficients in $\frac{1}{2} \sum \zeta'$ are integers. Reference to (41) shows that every x' , expressed as a function of the x , has integer coefficients. Thus the substitution (46), regarded as giving x' in terms of x , has integer

coefficients. It is symmetrical in x and x' , and the reciprocal substitution is therefore also integer. The substitutions are consequently both unimodular,* and all forms (i) are thus equivalent.

9.4. Forms of the three other types are readily converted into forms of type (i) if we first transform the sum of the first eight squares by the identity

$$\sum_{r=1}^8 a_r^2 = \sum_{r=1}^8 (a_r - s)^2 \quad \left(s = \frac{1}{4} \sum_{r=1}^8 a_r\right)$$

and then use the integer unimodular substitution

$$x'_6 = -\sum_{r=1}^8 x_r, \quad x'_r = x_r \quad (r \neq 6).$$

For, writing $\sum p = 2q+1$, we have

$$\text{in (ii)} \quad \begin{aligned} s &= \frac{1}{2}x_6 - \frac{1}{2}x'_6 + \left(\frac{1}{2}q + \frac{3}{4}\right)x_9, \\ a_r - s &= \xi_r + \frac{1}{2}x'_6 + \left(p_r - \frac{1}{2}q - \frac{3}{4}\right)x_9; \end{aligned}$$

$$\text{in (iii),} \quad \begin{aligned} s &= \frac{1}{2}x_6 - \frac{1}{2}x'_6 + \left(\frac{1}{2}q + \frac{1}{4}\right)x_9, \\ a_r - s &= \xi_r + \frac{1}{2}x'_6 + \left(p_r - \frac{1}{2}q - \frac{1}{4}\right)x_9; \end{aligned}$$

$$\text{in (iv)} \quad \begin{aligned} s &= \frac{1}{2}x_6 - \frac{1}{2}x'_6 + \left(\frac{1}{2}q + \frac{3}{4}\right)x_9, \\ a_r - s &= \xi_r + \frac{1}{2}x'_6 + \left(p_r - \frac{1}{2}q - \frac{1}{4}\right)x_9, \\ a_8 - s &= \xi_8 + \frac{1}{2}x'_6 + \left(p_8 - \frac{1}{2}q - \frac{3}{4}\right)x_9. \end{aligned}$$

In every case the coefficient of x_9 is of the form $P \pm \frac{1}{4}$ where P is an integer. When q is even, $p - \frac{1}{2}q - \frac{3}{4}$, $p - \frac{1}{2}q - \frac{1}{4}$ give respectively $P + \frac{1}{4}$, $P - \frac{1}{4}$; when q is odd, the correspondence is reversed. Thus in all cases the numbers of plus and minus signs are even, and all extreme forms are equivalent. I select as most convenient the form of type (iii) in which $p_1, \dots, p_7 = 0$, $p_8 = 1$, namely

$$\begin{aligned} \Phi_9 = & (x_2 + x_3 + x_4 + x_5 + \frac{1}{2}x_6)^2 + (x_1 + x_2 + \frac{1}{2}x_6)^2 + (x_1 + x_3 + \frac{1}{2}x_6)^2 + \\ & + (x_4 + \frac{1}{2}x_6)^2 + (x_5 + \frac{1}{2}x_6)^2 + (x_7 + \frac{1}{2}x_6)^2 + (x_8 + \frac{1}{2}x_6)^2 + \\ & + (x_7 + x_8 + x_9 + \frac{1}{2}x_6)^2 + x_9^2. \end{aligned} \quad (48)$$

10. Variation of x_1, \dots, x_9

When $n > 9$ and x_{10}, \dots, x_n are not all zero, we therefore consider

$$F = \Theta_9 + G_{10}(x_{10}, \dots, x_n),$$

where

$$\Theta_9 = \sum_{r=1}^8 (\xi_r + \frac{1}{2}x_6 + p_r \pm \theta_r)^2 + (\xi_9 + p_9 \pm \theta_9)^2.$$

* For, if M, M' are the moduli (determinants) of two reciprocal substitutions, $MM' = 1$, and M, M' are integers.

Here ξ_1, \dots, ξ_7 are those of (40), and now

$$\xi_8 = x_7 + x_8 + x_9, \quad \xi_9 = x_9.$$

If we write

$$\zeta_r = \xi_r + \frac{1}{2}x_8 \quad (r = 1, \dots, 8), \quad \zeta_9 = \xi_9 (= x_9),$$

then

$$x_7 + x_8 + \frac{1}{2}x_8 = \zeta_8 - \zeta_9,$$

and we can still use (41) to give x_1, \dots, x_8 in terms of ζ , provided only that ζ_8 is replaced by $\zeta_8 - \zeta_9$. The condition for integer x (when the ζ are integers) is therefore still that $\sum \zeta$ be even, the summation now being from ζ_1 to ζ_9 .

The condition that Θ_9 attain its over-all minimum $\sum \theta^2$, namely, that the equations $\zeta_r = -p_r$ ($r = 1, \dots, 9$) have integer solutions in x , is therefore that $\sum \zeta$ be even, and therefore that $\sum p$ be even; and, as before, x_8 is even. Accordingly, when $\sum p$ is odd, the over-all minimum is excluded, and then, when x_8 is even, the minimum of Θ_9 is

$$1 - 2\theta_0 + \sum_9 \theta^2,$$

where $\theta_0 = \max(\theta_1, \dots, \theta_8)$. I find it convenient to separate the case in which θ_9 is this greatest θ . Then, writing

$$\omega_1 = \sum_8 \theta^2 + (1 - \theta_9)^2, \quad \omega_2 = 1 - 2\theta_\alpha + \sum_8 \theta^2 + \theta_9^2,$$

where $\theta_\alpha = \max(\theta_1, \dots, \theta_8)$, we have, when x_8 is even,

$$\min \Theta_9 = \min(\omega_1, \omega_2).$$

10.1. When x_8 is odd, the over-all minimum of Θ_9 is

$$\sum_8 (\frac{1}{2} - \theta)^2 + \theta_9^2,$$

which is attained when the equations

$$\zeta_r = -p_r + \epsilon_r \quad (r = 1, \dots, 8), \quad \zeta_9 = -p_9$$

have integer solutions in x , the ϵ having the same meaning as before. The condition that $\sum \zeta$ be even then requires that $\sum \epsilon$ be odd since $\sum p$ is now odd. This over-all minimum of Θ_9 is therefore excluded when $\sum \epsilon$ is even, i.e. when there are even numbers of plus and minus signs before $\theta_1, \dots, \theta_8$. The minimum of Θ_9 is then one of

$$\omega_3 = \sum_8 (\frac{1}{2} - \theta)^2 + (1 - \theta_9)^2,$$

$$\omega_4 = 2\theta_\beta + \sum_8 (\frac{1}{2} - \theta)^2 + \theta_9^2,$$

where $\theta_\beta = \min(\theta_1, \dots, \theta_8)$. Thus, when x_8 is odd,

$$\min \Theta_9 = \min(\omega_3, \omega_4),$$

and so, for general x_8 ,

$$\min \Theta_9 = \min(\omega_1, \omega_2, \omega_3, \omega_4).$$

10.2. It remains to consider the greatest value of this minimum for variation of θ . We first remark that ω_2, ω_4 above differ from the ρ_1, ρ_2 of (42), (43) only in the presence of an added term θ_9^2 . Thus

$$\min(\omega_2, \omega_4) = \min(\rho_1, \rho_2) + \theta_9^2 = \Theta_8 + \theta_9^2$$

where Θ_8 is defined in § 9. There we have seen that $\Theta_8 \leq 1$ with equality only when (i) all $\theta = \frac{1}{4}$, (ii) all $\theta = \frac{1}{2}$, (iii) all $\theta = 0$, (iv) $\theta = 0, 0, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$ in some order. Thus

$$\min(\omega_2, \omega_4) \leq 1 + \theta_9^2,$$

with equality only under one of the four conditions just cited.

The values of ω_1, ω_3 corresponding to these values of θ are respectively

$$(i) \quad \omega_1 = \omega_3 = \min(\omega_1, \omega_3) = \frac{1}{2} + (1 - \theta_9)^2;$$

$$(ii) \quad \omega_1 = 2 + (1 - \theta_9)^2, \quad \omega_3 = (1 - \theta_9)^2;$$

$$(iii) \quad \omega_1 = (1 - \theta_9)^2, \quad \omega_3 = 2 + (1 - \theta_9)^2,$$

so that $\min(\omega_1, \omega_3) = (1 - \theta_9)^2$ in both (ii), (iii);

$$(iv) \quad \omega_1 = \omega_3 = \min(\omega_1, \omega_3) = 1 + (1 - \theta_9)^2.$$

Thus (iv) gives $\min(\omega_1, \omega_3)$ its greatest value, and, under this condition, we have

$$\begin{aligned} \min(\omega_1, \omega_2, \omega_3, \omega_4) &= \min\{1 + (1 - \theta_9)^2, 1 + \theta_9^2\} \\ &= 1 + \theta_9^2, \quad \text{since } \theta_9^2 \leq (1 - \theta_9)^2. \end{aligned}$$

This has its greatest value $\frac{5}{4}$ at $\theta_9 = \frac{1}{2}$. Accordingly $\min \Theta_9$ has its greatest value $\frac{5}{4}$ when $\theta_1, \dots, \theta_8$ are $0, 0, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$ in some order and $\theta_9 = \frac{1}{2}$. The condition on the signs of θ disappears because of the zero θ ; the condition on the parity of $\sum p$ is also indeterminate since, as usual, we replace $p_r - \frac{1}{2}$ by $(p_r - 1) + \frac{1}{2}$, effectively changing the parity of p_r . Thus

$$G_{10} = F - \Theta_9 \geq 2 - \frac{5}{4} = \frac{3}{4}.$$

10.3. When $n = 10$ and $x_{10} \neq 0$, we have

$$\begin{aligned} \min \Delta(G_{10}) &= \frac{3}{4}, & \Delta(F_{10}) &= \Delta_9 \Delta(G_{10}) = \Delta(G_{10}), \\ \Delta_{10} &= \min \Delta(G_{10}) = \frac{3}{4}, \end{aligned} \tag{49}$$

and so

$$\gamma_{10} = 2 \sqrt[10]{\frac{4}{3}}. \tag{50}$$

The extreme forms are

$$\sum \left\{ \xi_r + \frac{1}{2}x_8 + (p_r + \frac{1}{2})x_{10} \right\}^2 + \sum \left(\xi_s + \frac{1}{2}x_8 + p_s x_{10} \right)^2 + \\ + \left\{ x_9 + (p_9 + \frac{1}{2})x_{10} \right\}^2 + \frac{3}{4}x_{10}^2,$$

where r is taken over any four of the suffixes 1, ..., 8 and s over the other four. Two such forms with the same distribution of suffixes r, s are interchanged by the substitution

$$\zeta_r + p_r x_{10} \rightarrow \zeta_r + p'_r x_{10} \quad (r = 1, \dots, 9), \\ x_{10} \rightarrow x_{10},$$

which is integer and unimodular if $\sum p_r$ and $\sum p'_r$ have the same parity. If they have opposite parities, it is sufficient to change the sign of x_{10} in one form; this replaces p_r, p_s, p_9 by $-p_r-1, -p_s, -p_9-1$ and so changes the parity of an odd number of p and therefore of $\sum p$.

10.4. It remains to prove that two extreme forms with a different distribution of suffixes r, s but equal p are equivalent. Sufficiently I take all $p = 0$ and rearrange ζ_1, \dots, ζ_8 without change of x_9, x_{10} . The argument is essentially that of § 9.3. Since now

$$\zeta_8 = x_7 + x_8 + x_9 + \frac{1}{2}x_8,$$

we can still use the equations (41) to give x in terms of ζ , if we replace ζ_8 by $\zeta_8 - x_9$. We now have that the coefficients of x in $\frac{1}{2} \left(\sum_8 \zeta - x_9 \right)$ are integers. Any interchange of ζ_1, \dots, ζ_8 without change of x_9 gives

$$\frac{1}{2} \left(\sum_8 \zeta - x_9 \right) = \frac{1}{2} \left(\sum_8 \zeta' - x'_9 \right).$$

Since the expression on the right has integer coefficients in x' , so has the expression on the left.

Again it is still true, as in § 9.3, that in any sum or difference of two ζ the coefficients of the x are integers. Such sums and differences remain sums and differences after any interchange of ζ_1, \dots, ζ_8 . Thus, in any interchange of ζ_1, \dots, ζ_8 without change of x_9, x_{10} , every sum and difference of two ζ and also $\frac{1}{2} \left(\sum_8 \zeta - x_9 \right)$ are expressible in terms of the transformed x with integer coefficients. But, from the modified (41), x_1, \dots, x_8 are all expressible as sums of an even number of $\pm \zeta_1, \dots, \pm \zeta_8$ and possibly $\frac{1}{2} \left(\sum_8 \zeta - x_9 \right)$ and x_9 .

Hence any interchange of ζ_1, \dots, ζ_8 without change of x_9, x_{10} gives an integer substitution of x to x' . By symmetry the reciprocal

substitution is integer; and, by an earlier argument, both substitutions are therefore unimodular. Thus at length all the extreme forms are equivalent. For a convenient typical form I put all $p = 0$ and select the ξ_r, ξ_s to give

$$\begin{aligned}\Phi_{10} = & (x_2 + x_3 + x_4 + x_5 + \tfrac{1}{2}x_6)^2 + (x_1 + x_2 + \tfrac{1}{2}x_6)^2 + (x_1 + x_3 + \tfrac{1}{2}x_6)^2 + \\ & + (x_4 + \tfrac{1}{2}x_6 + \tfrac{1}{2}x_{10})^2 + (x_5 + \tfrac{1}{2}x_6 + \tfrac{1}{2}x_{10})^2 + \\ & + (x_7 + \tfrac{1}{2}x_6 + \tfrac{1}{2}x_{10})^2 + (x_8 + \tfrac{1}{2}x_6 + \tfrac{1}{2}x_{10})^2 + \\ & + (x_7 + x_8 + x_9 + \tfrac{1}{2}x_6)^2 + (x_9 + \tfrac{1}{2}x_{10})^2 + \tfrac{3}{4}x_{10}^2.\end{aligned}\quad (51)$$

10.5. In conclusion I wish to acknowledge my great obligation to Professor L. J. Mordell for his very generous and patient help and encouragement: in particular, I owe my interest in the problem to a lecture given by him to the Oxford Mathematical and Physical Society; and the later pages have gained much in clarity by a notable simplification at $n = 7$, due to him. I am also much indebted to Dr. K. M. Ollerenshaw for her energetic assistance.

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